Lecture 10  The Main Sequence

Last time

\[ \frac{dP}{dr} = -\frac{GM(r)}{r^2} \]

\[ L_r = -4\pi r^2 \frac{ac}{3\varepsilon p} \frac{dT}{dr} \]

\[ p = \left( \frac{\rho}{m_{\text{m}} n} \right) kT \]

Reference  Bahcall chapter 2

Estimate central pressure, temp. of Sun.

Consider crude three zone model

\[ \frac{dp}{dr} \approx \frac{GM_0}{r^2} \]  \( p \) from \( \frac{dp}{dr} = -\frac{GM_p}{r^2} \)

\[ p \approx \frac{GM_0}{r^2} \left< p \right> \]

\( \left< p \right> \approx 1 \text{g/cm}^3 \)

\( \approx 3 \times 10^5 \text{ dynes/cm}^2 \)
\[ T_c = \frac{P_c}{(\frac{\rho_c}{m_{\text{H}}})^k} = 10^7 \text{ K} \]

**Main Sequence Stars**

The equations of hydrostatic equilibrium lead to approximate scaling relations of different masses:

- \( L \propto R^2 \frac{T^4}{\langle \rho \rangle} \propto M \propto R^3 \)
- \( L \propto \frac{R^4}{\langle c_s \rangle^2} \propto \frac{M^4}{R^4} \)

Assume average opacity, approx. independent of \( T \).

EOS implies:

- \( T \propto \frac{P}{\rho} \propto \frac{M_0}{R P} \propto \frac{M}{R} \)
- \( L \propto \frac{R^4}{\langle c_s \rangle^2} \propto M^4 \propto M^3 \)

Measure masses of stars in binaries and we do find a sharp relation between \( L \) and \( M \); note slope is not exactly 3.
Hertzprung - Russell Diagrams

Plot luminosity vs surface temp. for a large number of stars.

Astronomers like logs

Bolometric Magnitude

\[ M_{\text{bol}} = 4.63 - 2.5 \log_{10} \left[ \frac{L}{L_\odot} \right] \]

Effective surface temp. IF star was a perfect black body of temp. \( T_{\text{eff}} \)

\[ L = 4\pi R^2 \sigma T_{\text{eff}}^4 \]

\( \sigma \) is Stephan Boltzman constant

Stars are not perfect black bodies but approx is not so bad.

Note: Sometimes uses stand ins for \( M_{\text{bol}} \) and \( T_{\text{eff}} \)

\[ M_V = 4.79 - 2.5 \log_{10} \left[ \frac{L_V}{L_{\odot}} \right] \]

energy radiated in \( V \) visual band

\( M_B \) involves radiation in \( B \) band

\( M_B \) involves radiation in Blue band

\( B - V \) = "color of star"

Hot stars are bluer and have large \( B \) then \( V \) luminosities, this gives a lower # for \( B - V \). For Sun \( M_V = 4.8 \), \( B - V = 0.6 \).
Gas clouds collapse to form protostars where they quickly burn all their deuterium.

\[ p + d \rightarrow ^3\text{He} + y \quad \text{E+M} \]

goes very much faster than weak

\[ p + p \rightarrow d + e^+ + v_e \]

These protostars have a large luminosity powered by gravitational contraction.

In millions of years they reach main sequence. Main diagonal band in HR diagram.

Stars spend most of their life on main sequence burning H to He.

Lifetime on main sequence is proportional to luminosity over mass (total amount of fuel).

\[ T_{\text{main seq}} \propto \frac{M}{L} \times \frac{M}{M^3} \times \frac{1}{M^2} \]

Low mass stars live much longer on main sequence. For Sun

\[ T_{\text{ms}} \approx 10^{10} \text{ yr} \]

Stars then become giants with cool expanded outer envelopes. Centre becomes hot and dense enough to finally burn
Helium via a three-body reaction

\[ \text{He} + \text{He} + \text{He} \rightarrow \text{C} + \gamma \]

Need rare 3-body rxn because no stable mass 5 or mass 8 nuclei.

Low mass giants \( \leq 8 \, \text{M}_\odot \)??

Eventually lose mass in a planetary nebula phase and become white dwarfs

High mass giants \( \geq 8 \, \text{M}_\odot \) burn all their fuel to Fe.

Then the core collapses during a spectacular supernova explosion.
Reaction Rates

Rate, number of reactions per unit volume per unit time between 1 and 2 is,

\[ R_{12} = \frac{n(1) n(2) \langle \sigma v \rangle_{12}}{(1 + S_{12})} \]

\[ n(1) = \text{# density of } 1 \]
\[ \langle \sigma v \rangle = \text{thermal average of cross section times relative velocity} \]

Units \[ [n] = \frac{1}{L^3} \quad \quad [\sigma v] = L^3 \text{time} \]

So \[ [R_{12}] = \frac{1}{L^3 \text{time}} \checkmark \]

Note if 1 and 2 are identical particles need to divide by two because you don't want to count the same pair 1,2 and 2,1 twice.

\[ \langle \sigma v \rangle = \frac{\int d^3v \; e^{-\frac{1}{2}mv^2/t} \; v \; \sigma(v)}{\int d^3v \; e^{-\frac{1}{2}mv^2/t}} \]

Average over a Boltzmann distribution

[Note, in principle should consider Fermi-Dirac or Bose-Einstein if system is very dense.]
\( \mu \) is reduced mass of two nuclei:

\[
\mu = \frac{M_1 M_2}{M_1 + M_2}
\]

Assume \( M_i \ll A_i A_{amu} \)

\( A_{amu} \) = atomic mass unit \( \approx 931 \text{ MeV} \)

\( = \frac{1}{12} \text{ of mass of } ^{12}C \)

Reduced number \( A \)

\[
A = \frac{A_1 A_2}{A_1 + A_2}
\]

\( M \ll A A_{amu} \)

\( E = \frac{1}{2} \mu v^2 \)

Need energy dependence of cross section.

1. Tunneling through Coulomb barrier depends drastically on energy. In a simple WKB approx. Square of wave function at origin depends on Gamow penetration factor

\[
f = \exp (-2\pi \gamma)
\]
\[ \eta = Z_1 Z_2 \frac{e^2}{h \nu} = \frac{Z_1 Z_2 \alpha}{\nu/c} \]

With \( \alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036} \)

2. Consider very low energy neutron reactions (no Coulomb barrier). The cross section can depend on the square of a length:

\[ \sigma \propto x^2 \propto \frac{\hbar^2}{p^2} \propto \frac{1}{E} \]

Cross section depends on square of de Broglie wave length. This also depends rapidly on \( E \) for very small \( E \) we expect

\[ \sigma \propto \frac{1}{E} \ e^{-2\pi \eta} \]

And define \( S(E) \) so that

\[ \sigma = \frac{S(E)}{E} e^{-2\pi \eta} \text{ or } \boxed{S(E) = \sigma(E) E e^{2\pi \eta}} \]
We expect $S(E)$ to depend much less on $E$ than $\sigma(E)$. $S(E)$ is a cross section or factor.