0] Consider neutrinos in thermal equilibrium at a temperature $T$ and zero chemical potential. Calculate the average neutrino energy $<E_\nu>$.
   
a) For a Maxwell Boltzmann distribution show

   $$<E_\nu> = 3kT$$

   b) Calculate $<E_\nu>$ for a Fermi Dirac distribution.

1] Gravitational binding energy of a neutron star.

  The gravitational binding energy (in Newtonian gravity) of a uniform sphere of mass $M$ and radius $R$ is

  $$BE = \frac{3}{5} \frac{GM^2}{R}$$

  a) Calculate $BE$ for a neutron star with $M = 1.4M_\odot$ and $R = 10km$,

  b) The gravitational mass of a neutron star determines the orbit of planets and binary stars. The baryon mass, $M_B$ is the amount of material that had to collapse to form the gravitational mass. Thus

  $$M_B = M + |BE|$$

  Calculate $M_B$ for this $1.4M_\odot$ star and calculate the number of baryons contained in $M_B$.

  c) Assume the binding energy $BE$ is radiated away in neutrinos of temperature $kT \sim 5MeV$ or from problem 0, $<E_\nu> \approx 15MeV$. Calculate the total # of neutrinos radiated and the number of $\nu$ radiated per baryon, part b), in the star.
2] Neutrino diffusion time in a supernova

a) Calculate the average baryon density in the neutron star. Use your baryon # from problem 1 (b) and divide by a volume \( \frac{4}{3} \pi R^3 \) with \( R = 10 \text{km} \). Show \( \rho \approx 3 \rho_o \) where the central baryon density of normal nuclei is \( \rho_o \approx 0.15 \text{Fm}^{-3} \).

b) The neutrino mean free path is

\[
\lambda = \frac{1}{\sigma \rho}
\]

Assume

\[
\sigma = \frac{G_F^2}{\pi}(g_\nu^2 + 3g_a^2)E_\nu^2
\]

with \( G_F \), the Fermi constant, \( g_\nu = 1 \) and \( g_a = 1.26 \). Let the temperature inside the supernova be \( kT \approx 10 \text{ MeV} \) so \( < E_\nu > \approx 30 \text{ MeV} \). Calculate \( \lambda \) assuming the average density from part a). Note the temperature inside is greater than the surface temperature.

c) The neutrinos will escape from the star in a time of order

\[
\tau \sim \frac{R^2}{\lambda c}
\]

This is the time to diffuse a distance \( R = 10 \text{km} \). Calculate \( \tau \) using \( \lambda \) from part (b). Note, the neutrino signal from a supernova will have a duration roughly characterized by \( \tau \). For example, the signal could be a decaying exponential with a decay constant of order \( \tau \). Compare with the \( \sim 10 \text{ seconds} \) over which \( \nu \) were detected from SN 1987a.


a) Consider pure hydrogen at low temperature. Calculate the threshold density for electron capture,

\[
e^- + p \rightarrow n + \nu_e
\]

b) Assume the hydrogen contains a small amount of \(^3\text{He}\). Calculate the threshold density for,
\[ e^- + ^3He \rightarrow ^3H + \nu_e \]

Note the mass of \(^3H\) is 18 keV above the mass of \(^3He\) and an electron.

4] The URCA process in neutron stars

Consider the reaction, 

\[ n \rightarrow p + e^- + \bar{\nu}_e, \]

followed by, 

\[ e^- + p \rightarrow n + \nu_e, \]

in the interior of a neutron star at low temperatures. Assume the energy of the emitted \(\nu_e\) or \(\bar{\nu}_e\) is very small compared to the neutron Fermi energy. Also assume a \(n\) very near its Fermi surface beta decays to a proton and an electron both very near their respective Fermi surfaces. Show that momentum conservation requires the electron fraction (electrons per baryon, \(Y_e\)) to be greater than 1/9. Note charge neutrality says the proton and electron Fermi momenta are equal so you can show

\[ Y_e = \frac{k_F^p}{(k_F^p + k_F^n)} \]

Here \(k_F^n, k_F^p\) are the neutron and proton Fermi momenta.