9/27/06

Lecture 14 Small Oscillations

For static solution all generalized forces vanish

$$Q_\sigma = -\left.\frac{\partial V}{\partial q_\sigma}\right|_{q_\sigma = q_\sigma^0} = 0$$

Expand coordinates and Lagrangian about equilibrium q_σ^0

$$q_\sigma = q_\sigma^0 + \eta_\sigma \qquad \dot{q}_\sigma = \dot{\eta}_\sigma$$

With η_σ small

$$T = \frac{1}{2}\sum_{\sigma\lambda} m_{\sigma\lambda}\,\dot{\eta}_\sigma\,\dot{\eta}_\lambda$$

$$m_{\sigma\lambda} \equiv \sum_i m_i \left.\frac{\partial x_i}{\partial q_\sigma}\right|_{q_\sigma^0} \left.\frac{\partial x_i}{\partial q_\lambda}\right|_{q^0} = m_{\lambda\sigma} = m^*_{\sigma\lambda}$$

$m_{\sigma\lambda}$ is a real symmetric matrix

$$V = V(q_1^0 + \eta_1, \ldots, q_n^0 + \eta_n)$$

$$\simeq \underbrace{V(q_1^0, \ldots, q_n^0)}_{V_0} + \sum_\sigma \eta_\sigma \left.\frac{\partial V}{\partial q_\sigma}\right|_{q_0}$$

$$+ \frac{1}{2}\sum_{\sigma\lambda} \eta_\sigma \eta_\lambda \left.\frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}\right|_{q_0}$$

At equilibrium $\partial V / \partial q_\lambda = 0$

$$V \cong V_0 + \frac{1}{2} \sum_{\sigma\lambda} v_{\sigma\lambda} \eta_\sigma \eta_\lambda$$

$$v_{\sigma\lambda} \equiv \left.\frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}\right|_{q^0} = v_{\lambda\sigma} = v^*_{\sigma\lambda}$$ real symmetric matrix

$$L = T - V = \frac{1}{2} \sum_{\sigma\lambda} (m_{\sigma\lambda} \dot\eta_\sigma \dot\eta_\lambda - v_{\sigma\lambda} \eta_\sigma \eta_\lambda) + V_0$$

Equations of motion $\frac{d}{dt} \frac{\partial L}{\partial \dot\eta_\sigma} - \partial L / \partial \eta_\sigma = 0$

$$\boxed{\sum_\lambda m_{\sigma\lambda} \ddot\eta_\lambda + v_{\sigma\lambda} \eta_\lambda = 0} \qquad \sigma = 1, \ldots, n$$

Note equations of motion are good to first order in η_σ while to get them needed to calculate L to 2^{nd} order in η_σ

Linear matrix equation. In general

$$\ddot\eta \propto \eta \Rightarrow \eta \sim \eta_0 e^{\pm i\omega t}$$

Consider one degree of freedom

$$m \ddot\eta + v \eta = 0$$

$$\ddot\eta = -\frac{v}{m} \eta$$

$$\eta = \mathrm{Re}\, \eta_0 e^{\pm i\omega t} \qquad \omega = \sqrt{\frac{v}{m}}$$

So for coupled problem guess that all coordinates oscillate with same frequency ω

$$\eta_{\sigma}(t) = \mathrm{Re}\, \eta_\sigma^c \, e^{i\omega t} \qquad \ddot{\eta}_\sigma = -\omega^2 \eta_\sigma$$

Here η_σ^c is a constant (time indep.) vector which contains information on the amplitude and relative phase of the different osc.

$$\text{(A)} \quad \sum_\lambda \left(V_{\sigma\lambda} - \omega^2 m_{\sigma\lambda} \right) \eta_\lambda^c = 0$$

This only has nontrivial solutions if

$$\det \left| V_{\sigma\lambda} - \omega^2 m_{\sigma\lambda} \right| = 0$$

This gives an n^{th} order polynomial in ω^2

Prove ω^2 is real

Multiply (A) by η_σ^{c*}

$$\sum_{\sigma\lambda} \eta_\sigma^{c*} \left(V_{\sigma\lambda} - \omega^2 m_{\sigma\lambda} \right) \eta_\lambda^c = 0$$

solve for ω^2

$$\omega^2 = \sum_{\sigma\lambda} V_{\sigma\lambda} \eta_\sigma^{c*} \eta_\lambda^c \Big/ \sum_{\sigma\lambda} m_{\sigma\lambda} \eta_\sigma^{c*} \eta_\lambda^c$$

Take complex conjugate

$$(\omega^2)^* = \sum_{\sigma\lambda} V_{\sigma\lambda}^* \eta_\sigma^c \eta_\lambda^{c*} \Big/ \sum_{\sigma\lambda} m_{\sigma\lambda}^* \eta_\sigma^c \eta_\lambda^{c*}$$

but

$$V_{\sigma\lambda}^* = V_{\sigma\lambda} = V_{\lambda\sigma} \qquad m_{\sigma\lambda}^* = m_{\lambda\sigma}$$

$$(\omega^2)^* = \sum_{\sigma\lambda} v_{\lambda\sigma}\, \eta_\sigma^0\, \eta_\lambda^{0*} \Big/ \sum_{\sigma\lambda} m_{\lambda\sigma}\, \eta_\sigma^0\, \eta_\lambda^{0*}$$

now let $\lambda \to \sigma$ $\quad \sigma \to \lambda$

$$= \sum_{\sigma\lambda} v_{\sigma\lambda}\, \eta_\lambda^0\, \eta_\sigma^{0*} \Big/ \sum_{\sigma\lambda} m_{\sigma\lambda}\, \eta_\lambda^0\, \eta_\sigma^{0*}$$

$$= \omega^2$$

Therefore ω^2 is real.

Stable if all $\omega^2 > 0$
Unstable if any $\omega^2 < 0$