Lecture 14: Small Oscillations

For static solution all generalized forces vanish:

$$Q_0 = -\frac{\partial V}{\partial \theta_0} \bigg|_{\theta_0 = \theta_0^0} = 0$$

Expand coordinates and Lagrangian about equilibrium $$\theta_0^0$$:

$$\theta_0 = \theta_0^0 + \eta_0$$

With $$\eta_0$$ small:

$$T = \frac{1}{2} \sum \limits_{i} m_i \dot{\eta}_i \eta_i$$

$$m_{i\chi} = \sum \limits_{i} m_i \frac{\partial x_i}{\partial \theta_0^0} \frac{\partial x_{i\chi}}{\partial \theta_0^0} = m \lambda_{i\chi}$$

$$m_{i\chi}$$ is a real symmetric matrix

$$V = V(\theta_0^0 + \eta_0^1, \ldots, \theta_0^0 + \eta_0^n)$$

$$= V(\eta_0^1, \ldots, \eta_0^n) + \sum \limits_{i} \eta_0^i \frac{\partial V}{\partial \theta_0^0} \bigg|_{\theta_0^0}$$

$$+ \frac{1}{2} \sum \limits_{i} \eta_0^i \eta_0^i \frac{\partial^2 V}{\partial \theta_0^0 \partial \theta_0^0} \bigg|_{\theta_0^0}$$
At equilibrium \( \partial V/\partial q_k = 0 \)

\[ V = V_0 + \frac{1}{2} \sum_{k \neq l} V_{kl} \eta_k \eta_l \]

\[ V_{kl} = \frac{\partial^2 V}{\partial q_k \partial q_l} \]

\[ V_{kl} = V_{lk} \quad \text{real matrix} \]

\[ L = T - V = \frac{1}{2} \sum_{k \neq l} \left( m \dot{\eta}_k \dot{\eta}_l - V_k \eta_k \eta_l \right) + V_0 \]

Equations of motion \[ \frac{d^2}{dt^2} \eta_k + V_{kk} \eta_k = 0 \]

Note: Equations of motion are good to first order in \( \eta \) while to get them needed to calculate \( L \) to 2nd order in \( \eta \).

Linear matrix equation. In general

\[ \eta' = \eta - \eta_0 e^{-t} \]

Consider one degree of freedom

\[ m \ddot{\eta} + V \eta = 0 \]

\[ \ddot{\eta} = -\frac{V}{m} \eta \]

\[ \eta = \Re \eta_0 e^{i \omega t} \]

\[ \omega = \sqrt{\frac{V}{m}} \]

So for coupled problem guess that all coordinates oscillate with same frequency \( \omega \).
\[ \eta_{6t} = \Re \eta_6^* e^{i \omega t} \quad \eta_6 = -\omega^2 \eta_6 \]

Here, \( \eta_6^* \) is a constant (time independent) vector which contains information on the amplitude and relative phase of the different oscillators.

(4) \[ \sum (V_\Omega x - \omega^2 m_\Omega x) \eta_6^* = 0 \]

This only has nontrivial solutions if

\[ \det \left| V_\Omega x - \omega^2 m_\Omega x \right| = 0 \]

This gives an \( n \)th order polynomial in \( \omega^2 \)

Prove \( \omega^2 \) is real

Multiply (4) by \( \eta_6 \)

\[ \sum \eta_6^* (V_\Omega x - \omega^2 m_\Omega x) \eta_6 = 0 \]

Solve for \( \omega^2 \)

\[ \omega^2 = \sum \frac{V_\Omega x \eta_6^* \eta_6}{m_\Omega x} \]

Take complex conjugate

\[ (\omega^2)^* = \sum \frac{V_\Omega x \eta_6^* \eta_6^*}{m_\Omega x} \]

but

\[ V_\Omega x = V_\Omega x = V_\Omega \]

\[ m_\Omega x = m_\Omega \]

\[ m_\Omega^* = m_\Omega \]
\((\omega^2)^* \leq \sum \eta_0 \eta_x \lambda / \sum \eta_0 \eta_x \lambda \)

Now let \( \lambda \geq 6 \), \( \lambda \to \lambda \)

\[ = \sum \eta_0 \eta_x \lambda \eta_0^* / \sum \omega_0 \eta_0 \eta_0^* \]

\[ = \omega^2 \]

Therefore \( \omega^2 \) is real.

Stable if all \( \omega^2 > 0 \)

Unstable if any \( \omega^2 < 0 \)