No class 9/22 ν workshop 9/20/06

Lecture 11 Forces of Constraint

Given n degrees of freedom and k constraints

(1) Chose $n-k$ independent generalized coordinates and minimize

$$A = \int_{t_1}^{t_2} dt\, L[q_\sigma(t), \dot{q}_\sigma(t), t]$$

(2) Or chose n coordinates and minimize

$$I = \int dt \left[L + \sum_{i=1}^{k} \lambda_i f_i(q_1 \cdots q_n, t) \right]$$

where constraints $f_i(q_1 \cdots q_n, t) = c_i \quad i = 1, \ldots k$ have been included with Lagrange multipliers λ_i

Minimizing I gives

$$\delta I = 0 = \int dt \sum_{\sigma=1}^{n} \delta q_\sigma \left[\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \sum_{i=1}^{k} \lambda_i \frac{\partial f_i}{\partial q_\sigma} \right]$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_i \lambda_i \frac{\partial f_i}{\partial q_\sigma} \qquad \sigma = 1, \ldots, n$$

and

$$f_i = c_i \qquad i = 1, \ldots, k$$

This is a set of $n+k$ equations in $n+k$ unknowns (n q_σ and k λ_i)

Choice (2) allows calculation of forces of constraint

Remember original derivation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma \leftarrow \text{generalized force}$$

Now

$$Q_\sigma = -\frac{\partial V}{\partial q_\sigma} + Q^r_\sigma$$

$$Q^r_\sigma = \sum_i \lambda_i \frac{\partial f_i}{\partial q_\sigma}$$ reaction force or force of constraint

Need to calculate reaction force for example if you have a weak string which constrains a particle till it breaks or if you are constrained to be on or above a surface ... [Force of constraint goes to zero when you leave surface.]

Example Pendulum

$$L = \frac{1}{2} m(\dot{r}^2 + r\dot{\theta}^2) + mgr\cos\theta$$

Constraint $f_1(r) = r = l$

$$\frac{\partial f_1}{\partial r} = 1 \qquad \frac{\partial f_1}{\partial \theta} = 0$$

Unknowns $r(t)$, $\theta(t)$, $\lambda_1(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda_1 \frac{\partial f_1}{\partial r}$$

(1) $m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda_1$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial f_1}{\partial \theta} = 0$$

(2) $$\frac{d}{dt}\left(m r^2 \dot\theta\right) + m g r \sin\theta = 0$$

(3) $$f_1 = l = r$$

Three equations in 3 unknowns

From (3) $\ddot r = \dot r = 0$ so

$$\lambda_1 = -m g \cos\theta - m r \dot\theta^2$$

and

$$\boxed{\ddot\theta = -\frac{g}{l} \sin\theta}$$

To interpret reaction force $Q_r = \lambda_1 \frac{\partial f_1}{\partial r} = \lambda_1$

Carry out a virtual displacement of r and compute work

$$\delta W = Q_r \, \delta r$$

Tension in string pulls against δr so

$$\delta W = -\tau \, \delta r$$

$\Rightarrow$ Tension $\boxed{\tau = -Q_r = m g \cos\theta + m l \dot\theta^2}$

Can compare this to τ_{max} to see if string breaks

Generalized momenta and the Hamiltonian

Read over cylinder on cylinder problem

Define <u>generalized momentum</u> or canonical momentum

$$p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$$

p_σ is momentum conjugate to q_σ

Lagrange's equations $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = 0$

$$\dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma}$$

If some generalized coordinate q_σ does not appear explicitly in Lagrangian than

$$\frac{\partial L}{\partial q_\sigma} = 0 \implies p_\sigma = \text{constant}$$

Such coordinates are called <u>cyclic</u>

Generalized momenta corresponding to cyclic coordinates are conserved

Symmetries and Conserved Quantities

If system is invariant under some continuous transformation, then T, V are unchanged by altering the corresponding coordinate q_σ so

$$\frac{\partial L}{\partial q_\sigma} = 0 \implies p_\sigma \text{ is conserved}$$

The existance of a continuous symmetry implies a conserved generalized momentum

Example Motion in a central pot.

$$L = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \dot{p}_\phi = 0$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \text{angular momentum}$$

$$\frac{d}{dt}(m r^2 \dot{\phi}) = 0$$

Hamiltonian

$$H \equiv \sum_\sigma p_\sigma \dot{q}_\sigma - L$$

This is important in mechanics because

(1) If L does not depend explicitly on time, then the Hamiltonian is a constant of the motion

$$dH = \sum_\sigma \left[p_\sigma d\dot{q}_\sigma + \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right] - \frac{\partial L}{\partial t} dt$$

divide by dt

$$\frac{dH}{dt} = \sum_\sigma \dot{q}_\sigma \frac{dp_\sigma}{dt} - \frac{\partial L}{\partial q_\sigma} \dot{q}_\sigma$$

$$= \sum_\sigma \dot{q}_\sigma \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} \right] \equiv 0$$

by Lagrange's equations $\Rightarrow$ H is a constant of motion

(2) If the pot. is time independent and the constraint are time independent then the hamiltonian is the energy.

$$x_i = x_i(q_1 \ldots q_{n-k}, t)$$

$$\dot{x}_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma}\dot{q}_\sigma + \cancel{\frac{\partial x_i}{\partial t}} \rightarrow 0$$

for time indep. constraints

$$T = \frac{1}{2}\sum_i m_i \dot{x}_i^2 = \frac{1}{2}\sum_{\sigma\lambda}\sum_i \left(m_i \frac{\partial x_i}{\partial q_\sigma}\frac{\partial x_i}{\partial q_\lambda}\right)\dot{q}_\sigma \dot{q}_\lambda$$

Define

$$m_{\sigma\lambda} \equiv \sum_i m_i \frac{\partial x_i}{\partial q_\sigma}\frac{\partial x_i}{\partial q_\lambda} = m_{\lambda\sigma}$$

$$T = \frac{1}{2}\sum_{\sigma\lambda} m_{\sigma\lambda}(q)\,\dot{q}_\sigma \dot{q}_\lambda$$

no $\dot{q}$

So

$$\sum_\sigma p_\sigma \dot{q}_\sigma = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = \sum_\sigma \frac{\partial T}{\partial \dot{q}_\sigma}\dot{q}_\sigma$$

given $\frac{\partial V}{\partial \dot{q}_\sigma} = 0$ $\qquad \frac{\partial T}{\partial \dot{q}_\sigma} = \sum_\lambda m_{\sigma\lambda}\dot{q}_\lambda$

$$\sum_\sigma p_\sigma \dot{q}_\sigma = \sum_{\lambda\sigma} m_{\sigma\lambda}\dot{q}_\lambda \dot{q}_\sigma = 2T$$

$$H = \sum_\sigma p_\sigma \dot{q}_\sigma - L = 2T - T - -V$$

$$\boxed{H = T + V = E}$$

True if $\frac{\partial L}{\partial t} = 0$ and constraints are time independent