Last time: bead on a rotating hoop

\[ x = a \cos(wt + \cos(wt + \phi)) \]
\[ y = a \sin(wt + \sin(wt + \phi)) \]
\[ V = 0 \]
\[ T = \frac{1}{2} m (x^2 + y^2) \]

Always start with \[ T = \frac{1}{2} m (x^2 + y^2) \]
\[ \dot{x} = -a \left[ \cos(wt + \phi) \sin(wt + \phi) \right] \]
\[ T = L = \frac{1}{2} m a^2 \left[ w^2 + (w + \dot{\phi})^2 + 2w (w + \dot{\phi}) \cos \phi \right] \]
\[ \frac{dL}{d\phi} = m a^2 \left[ (w + \dot{\phi}) + w \cos \phi \right] \]
\[ \frac{d}{dt} \frac{dL}{d\phi} = m a^2 \left[ \ddot{\phi} - w \sin \phi \dot{\phi} \right] \]

\[ \frac{dL}{d\phi} = -ma^2 w (w + \dot{\phi}) \sin \phi \]
\[ ma^2 \left[ \ddot{\phi} - w \sin \phi \dot{\phi} + w^2 \sin \phi + w \dot{\phi} \sin \phi \right] = 0 \]
\[ \ddot{\phi} = -w^2 \sin \phi \]

Pendulum equation \[ \frac{g}{L} \rightarrow w^2 \]
Calculus of Variations

Redrive Lagrange's equations using Hamilton's Principle → Minimize Action

First introduce calculus of variations as a tool to minimize action.

Example: Brachistochrone Problem

Distance along curve $= \int \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx$

Velocity $\frac{1}{2} mv^2 = mgy \implies v = \sqrt{2gy}$

$t_{12} = \int_{1}^{2} \left[ \frac{1 + y'^2}{2gy} \right]^{\frac{1}{2}} dx$

In general have a functional $I = I[y(x), y'(x)]$

Minimize $I_{12}$

$I = \int_{x_1}^{x_2} \phi [y(x), y'(x), x] dx$

For our problem $\phi = \left[ \frac{1 + y'^2}{2gy} \right]^{\frac{1}{2}}$
Consider variation
\[ y(x) \rightarrow y^*(x) = y(x) + \delta y(x) \]
With boundary conditions
\[ \delta y(x_1) = \delta y(x_2) = 0 \]
Since we know the end points of wire
\[ y'(x) = y'(x) + \delta y'(x) \]
So \[ I = \int_{x_1}^{x_2} \phi \left( y, y', x \right) \, dx - \int_{y_1}^{y_2} \phi \left( y, y', x \right) \, dy \]
Taylor expand
\[ \phi \left( y, y', x \right) \approx \phi \left( y, y', x \right) + \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \delta y' \]
So \[ I = \int_{x_1}^{x_2} \left[ \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \frac{dy}{dx} \delta y' \right] \, dx \]
What we mean by \( \delta y' \)
So \[ I = 0 \] for minimum path.
Integrate 2nd term by parts
\[ \int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'} \frac{dy}{dx} \delta y' \, dx = \frac{\partial \phi}{\partial y'} \left. \delta y' \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left[ \frac{\partial \phi}{\partial y'} \right] \, dx \]
but boundary conditions
\[ \delta y(x_1) = \delta y(x_2) = 0 \]
\[ S_I = \int_a^b x^2 \left[ \frac{\partial \Phi}{\partial y} - \frac{\partial}{\partial x} \frac{\partial \Phi}{\partial y} \right] \, dx \, dy = 0 \]

For: \( \Phi \). Int. to require
\[ \frac{d}{dx} \left[ \frac{\partial \Phi}{\partial y} \right] - \frac{\partial \Phi}{\partial y} = 0 \]

to minimize \( I \)

**Hamilton's Principle**

LaGrangie\'s equations of motion minimize the action
\[ A = \int_{t_1}^{t_2} dt \left[ L \left[ q_i(t), \dot{q}_i(t), t \right] \right] \]

\[ S = \int_{t_1}^{t_2} dt \left[ L \left[ q_i(t), \dot{q}_i(t), t \right] \right] = 0 \]

Taylor expand
\[ = \int_{t_1}^{t_2} dt \sum_{n=1}^{\infty} \frac{d^n}{dt^n} L \left[ q_i(t), \dot{q}_i(t), t \right] \]

Integrate and term by parts \( S q_i = \int \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \, dt \)

B.C. \( q_i(t_1) = q_i(t_2) = 0 \)

\[ 0 = \int_{t_1}^{t_2} dt \left[ \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \, dt \]

If, all \( \dot{q}_i \) are independent then
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 
\]

\[\Rightarrow \text{Lagrange equations}\]

If some of the \( q_i \) are not independent because of holonomic constraints

\[F_j(q_1, \ldots, q_n; t) = c_j, \quad j = 1, \ldots, k\]

1. Could choose any set of \( n-k \) independent coordinates and for these

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0\]

2. Instead include constraints with Lagrange multipliers

\[\sum_{j=1}^{k} \lambda_j \frac{\partial F_j}{\partial q_i} = 0, \quad j = 1, \ldots, k\]

\[c = \sum_{i=1}^{n} \left( \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i} \right) + \sum_{j=1}^{k} \lambda_j \frac{\partial F_j}{\partial q_i} \left. \right|_{t_i, q_i} \]

and \[\sum_{j=1}^{k} \lambda_j \frac{\partial F_j}{\partial q_i} \] into action and choose \( \lambda_j \) so that \( F_j = 0 \)

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{k} \lambda_j \frac{\partial F_j}{\partial q_i}, \quad i = 1, \ldots, n\]

\[F_j(q_1, \ldots, q_n; t) = c_j, \quad j = 1, \ldots, k\]

Set of \( n+k \) equations in \( n+k \) unknowns