P521 Lecture 10

9/18/00

Last time Bead on a rotating hoop

$$x = a(\cos\omega t + \cos(\omega t+\theta))$$
$$y = a(\sin\omega t + \sin(\omega t+\theta))$$
$$V = 0$$
$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

always start with $T = \frac{1}{2} m(\dot{x}^2+\dot{y}^2)$

$$\dot{x} = -a\left[\omega\sin\omega t + (\omega+\dot\theta)\sin(\omega t+\theta)\right]$$

$$T = L = \tfrac{1}{2} m a^2\left[\omega^2 + (\omega+\dot\theta)^2 + 2\omega(\omega+\dot\theta)\cos\theta\right]$$

$$\frac{\partial L}{\partial\dot\theta} = m a^2\left[(\omega+\dot\theta) + \omega\cos\theta\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = m a^2\left[\ddot\theta - \omega\sin\theta\,\dot\theta\right]$$

$$\frac{\partial L}{\partial\theta} = -m a^2\,\omega(\omega+\dot\theta)\sin\theta$$

$$m a^2\left[\ddot\theta - \omega\sin\theta\,\dot\theta + \omega^2\sin\theta + \omega\dot\theta\sin\theta\right] = 0$$

$$\boxed{\ddot\theta = -\omega^2\sin\theta}$$

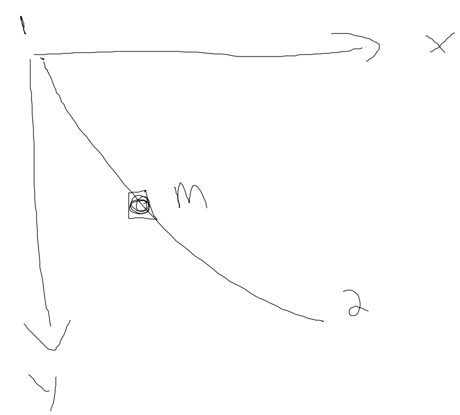
Pendulum equation $\frac{g}{l} \to \omega^2$

Calculus of Variations

Rederive Lagrange's equations using Hamilton's Principle $\rightarrow$ Minimize Action

First introduce calculus of variations as a tool to minimize action.

Example: Brachistochrone Problem



Bead falls on wire from 1 to 2. What is shape of wire which minimizes transit time?

Distance along curve $= \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$

Velocity $\frac{1}{2}mv^2 = mgy \Rightarrow v = 2gy^{1/2}$

$$t_{12} = \int_1^2 \left[\frac{1+y'^2}{2gy}\right]^{1/2} dx$$

In general have a functional $\phi \equiv \phi[y(x), y'(x), x]$

Minimize

$$I = \int_{x_1}^{x_2} \phi[y(x), y'(x), x]\, dx$$

For our problem $\phi = \left[\frac{1+y'^2}{2gy}\right]^{1/2}$

Consider variation

$$y(x) \to Y(x) = y(x) + \delta y(x)$$

With boundary conditions

$$\delta y(x_1) = \delta y(x_2) = 0$$

since we know the end points of wire

$$Y'(x) = y'(x) + \delta y'(x)$$

$$\delta I = \int dx \{ \phi[Y, Y', x] - \phi[y, y', x] \}$$

Taylor expand

$$\phi[Y, Y', x] \simeq \phi(y, y', x) + \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \delta y'$$

$$\delta I = \int_{x_1}^{x_2} dx \left[\frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \frac{d}{dx}(\delta y) \right] = 0$$

what we mean by $\delta y'$

$\delta I = 0$ for minimum path.

Integrate 2nd term by parts

$$\int_{x_1}^{x_2} dx \, \frac{\partial \phi}{\partial y'} \frac{d}{dx} \delta y = \delta y \frac{\partial \phi}{\partial y'} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \left[\frac{d}{dx} \left(\frac{\partial \phi}{\partial y'} \right) \right]$$

but boundary conditions

$$\delta y(x_1) = \delta y(x_2) = 0$$

$$\delta I = \int_{x_1}^{x_2} dx \left[\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'}\right] \delta y = 0$$

For arbitrary δy require

$$\frac{d}{dx}\left[\frac{\partial \phi}{\partial y'}\right] - \frac{\partial \phi}{\partial y} = 0$$

to minimize I

Hamilton's Principle

Lagrange's equations of motion minimize the action

$$A \equiv \int_{t_1}^{t_2} dt \; L[q_\sigma(t), \dot{q}_\sigma(t), t]$$

$$\delta \int_{t_1}^{t_2} dt \; L[q_\sigma(t), \dot{q}_\sigma(t), t] = 0$$

Taylor expand

$$= \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \left[\frac{\partial L}{\partial q_\sigma}\delta q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma}\delta \dot{q}_\sigma\right] dt$$

Integrating 2nd term by parts $\delta \dot{q}_\sigma = \frac{d}{dt}\delta q_\sigma$

B.C. $\delta q_\sigma(t_1) = \delta q_\sigma(t_2) = 0$

$$0 = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\right] \delta q_\sigma$$

If all δq_σ are independent then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Lagrange equations

If some of the δq_σ are not independent because of holonomic constraints

$$f_j(q_1 \ldots q_n, t) = c_j \qquad j = 1, \ldots, k$$

(1) Could chose any set of $n-k$ independent coordinates and for these

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = 0$$

(2) Instead include constraints with Lagrange multipliers

$$\delta f_j = \sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma = 0 \qquad j = 1, \ldots, k$$

$$0 = \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \delta q_\sigma \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} \right) dt = 0$$

add $\sum_j \lambda_j \delta f_j$ into action and chose λ_j so ~~that~~ $f_j = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \qquad \sigma = 1, \ldots n$$

$$f_j(q_1 \ldots q_n, t) = c_j \qquad j = 1, \ldots, k$$

Set of $n+k$ equations in $n+k$ unknowns