9/15/00

P521 Lecture 9 Lagrangian Dynamics Cont.

Last time Lagrangian Dynamics $\Rightarrow$ Rewrite Newton's laws (a) in generalized coordinates (b) without explicit forces of constraint

Holonomic constraints $F_j(x_1, x_2, \dots x_n, t) = 0$
$j = 1, \dots, k$

Virtual displacement small displacement consistent with constraints

D'alembert's principle Forces of constraint do no work under a virtual displacement.

Newton's law $\sum_i (F_i + R_i - \dot{p}_i)\, \delta x_i = 0$

applied force, reaction force

$$\sum_i R_i \delta x_i = 0 \Rightarrow \boxed{\sum_i (F_i - \dot{p}_i)\, \delta x_i = 0} \quad (*)$$

$$\sum_{i=1}^{n} F_i \delta x_i = \sum_{\sigma=1}^{n-k} Q_\sigma \delta q_\sigma$$

Generalized Force $Q_\sigma \equiv \sum_i F_i \frac{\partial x_i}{\partial q_\sigma}$

$$\sum_i \dot{p}_i \delta x_i = \sum_\sigma \sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

$$= \sum_\sigma \sum_i m_i \Big[\underbrace{\frac{d}{dt}\Big(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma}\Big)}_{A} - \underbrace{\dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma}}_{B} \Big] \delta q_\sigma$$

$$x_i = x_i(q_1, q_2 \cdots q_{n-k}, t)$$

$$\dot{x}_i = \frac{d}{dt} x_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma + \left.\frac{\partial x_i}{\partial t}\right|_{\text{fixed } q_\sigma}$$

so $\dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \dfrac{\partial x_i}{\partial q_\sigma}$

$$\sum_i m_i \frac{d}{dt}\left[\dot{x}_i \frac{\partial x_i}{\partial q_\sigma}\right] = \sum_i m_i \frac{d}{dt} \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$$

A $$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\sigma} \sum_i \frac{1}{2} m_i \dot{x}_i^2$$

B $$= \sum_i m_i \dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_\sigma}$$

$$= \frac{\partial}{\partial q_\sigma} \sum_i \frac{1}{2} m_i \dot{x}_i^2$$

Call $T \equiv \sum_i \frac{1}{2} m_i \dot{x}_i^2$

$$\sum_i \dot{p}_i \, \delta x_i = \sum_\sigma [\text{A} - \text{B}] \, \delta q_\sigma$$

$$= \sum_\sigma \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_\sigma} T - \frac{\partial}{\partial q_\sigma} T\right] \delta q_\sigma$$

So (*) becomes

$$\sum_\sigma \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_\sigma} T - \frac{\partial}{\partial q_\sigma} T - Q_\sigma\right] \delta q_\sigma = 0$$

or because all the δq_σ are independent

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_\sigma} T - \frac{\partial}{\partial q_\sigma} T - Q_\sigma = 0 \qquad \sigma = 1, \dots n-k$$

For a conservative force

$$F_i = -\frac{\partial}{\partial x_i} V$$

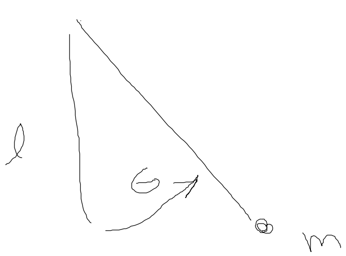
$$Q_\sigma = -\sum_i \frac{\partial}{\partial x_i} V \frac{\partial x_i}{\partial q_\sigma} = -\frac{\partial}{\partial q_\sigma} V$$

So define $L = T - V$

$$\boxed{\frac{d}{dt}\frac{\partial}{\partial \dot{q}_\sigma} L - \frac{\partial}{\partial q_\sigma} L = 0} \qquad \sigma = 1, \dots n-k$$

Note $\frac{\partial V}{\partial \dot{q}_\sigma} \equiv 0$

<u>Examples</u> Pendulum



$$T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$= \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

but $\dot{r} = 0 \quad r = l$

$$T = \tfrac{1}{2} m l^2 \dot{\theta}^2$$

$$V = -mgl\cos\theta + \text{const}$$

$$L = \tfrac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

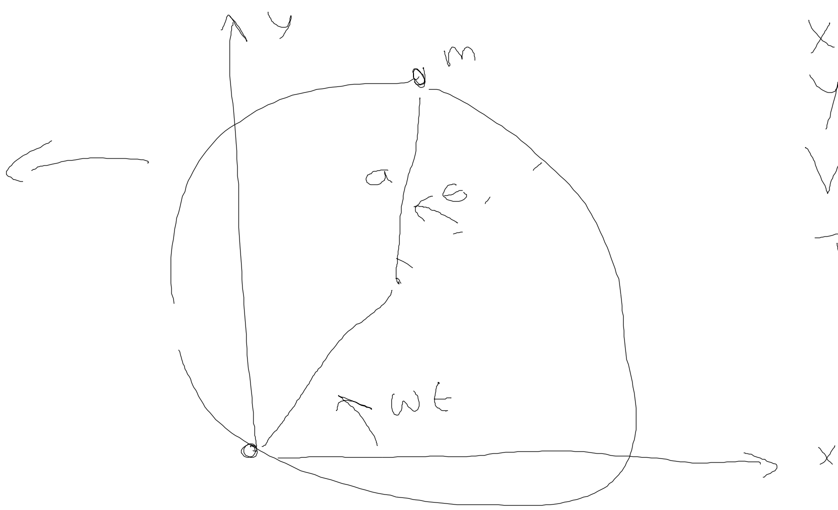
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m l^2 \ddot\theta \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$m l^2 \ddot\theta + mgl\sin\theta = 0$$

$$\Rightarrow \boxed{\ddot\theta = -\frac{g}{l}\sin\theta}$$

Never needed tension in string

<u>Bead on a rotating hoop</u>



$x = a\cos\omega t + a\cos(\omega t + \theta)$
$y = a\sin\omega t + a\sin(\omega t + \theta)$
$V = 0$
$T = \frac{1}{2} m(\dot x^2 + \dot y^2)$

Always start with $T = \frac{1}{2} m(\dot x^2 + \dot y^2)$

$\dot x = -a\omega\sin\omega t - a(\omega + \dot\theta)\sin(\omega t + \theta)$
etc.

$$T = L = \frac{1}{2} m a^2\left[\omega^2 + (\omega + \dot\theta)^2 + 2\omega(\omega + \dot\theta)\cos\theta\right]$$

Note kinetic energy function of both $\dot\theta$ and $\theta \Rightarrow$ For generalized coordinates expect q_σ to appear in kinetic energy.

$$\frac{d}{dt}\left[\frac{1}{2} m a^2 \left(\left[2(\omega+\dot\theta)\right] + 2\omega\cos\theta\right)\right]$$

$$= \frac{1}{2} m a^2 \left[-2\omega(\omega+\dot\theta)\sin\theta\right] = 0$$

or

$$\ddot\theta + \omega^2 \sin\theta = 0$$

looks like a pendulum