P521 Lecture 8

9/13/00

Go to colloquium today - B. Heckel!
Start reading Chapter 3 Lagrangian Dynamics

Reformulate Newton’s Laws
a) Using only independent degrees of freedom
b) Incorporating constraints without complicated forces of constraint

Example: bead on a wire. Start with N Laws in 3 dim
i) 3 degrees of freedom X, Y, Z
ii) Complex forces on bead stays on wire

Alternatively, only need one independent degree of freedom — for example displacement of bead along wire

Answer: Lagrangian Dynamics

$L = T - V$ note minus sign

Newton’s Laws are equivalent to

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Here $\{q_i\}$ are a set of generalized coordinates

Constraints

Consider N particles with $n=3N$ degrees of freedom

$$i = (1,2,3)(4,5,6)\ldots(n-2,n-1,n)$$
Holonomic Constraints

"Simple" explicit constraints that can be written as an equation

\[ F_j(x_1, x_2, \ldots, x_{nj}, t) = 0 \]

- Assume we have \( k \) such constraints
- For a bead on a wire \( k \) would be 2

Example: nonholonomic constraint - particle constrained to move on or above sphere.

If particle had to stay on sphere - holonomic constraint.

Generalized Coordinates

System with \( n \) degrees of freedom under \( k \) holonomic constraints has \( n - k \) independent degrees of freedom.

Let \( q_1, q_2, \ldots, q_{n-k} \) be any set of independent generalized coordinates that completely specify the configuration of the system.

\[ x_i = x_i(q_1, \ldots, q_{n-k}, t) \]

\[ x_n = x_n(q_1, \ldots, q_{n-k}, t) \]

Example: \( f_i \), bead on wire \( n = 3 \), \( k = 2 \) and \( q_i \), could be displacement of bead along wire, shape of wire gives \( x(q_i) \). z \( (q_i) \).
Virtual Displacement

In infinitesimal instantaneous displacement consistent with constraints.

Example: small motion of bead along wire.

\[ \delta X_i = \sum_{k=1}^{n-K} \delta x_i \delta g_k \]

\[ \delta x_i = \frac{\delta g_k}{3g_k} \]

Change in coordinates from equations giving coordinates from generalized coordinates.

D'alembert's principle

The reaction force (forces of constraint) do no work under a virtual displacement.

Example: bead on a wire. Virtual displacement is along wire. Force of constraint is 1 to wire to keep bead on wire. Force 1 to displacement does no work.

\[ P_i = F_i + R_i \]

\[ \delta P_i = \delta F_i + \delta R_i \]

\[ \delta (F_i + R_i) \delta x_i = 0 \]

\[ \sum_{i=1}^{n-K} \delta x_i = 0 \]

Given that \( F_i + R_i = 0 \) for all.

D'alembert's principle: \( \sum R_i \delta x_i = 0 \)
\[ \sum (F_i - p_i) \delta x_i = 0 \]

Note all the \( x_i \) are not in general independent. Need to change \( x, y, z \) in a controlled way to keep bond or wire.

Rewrite in terms of generalized coordinates where all \( x_i \) are independent.

Start with 1st term

\[ SW = \sum F_i \delta x_i \]

\[ SW = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} F_{ij} \left( \frac{\partial x_i}{\partial q_j} \right) \right] \delta q_j \]

Used \( \delta x_i = \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} \delta q_k \)

Generalized Force

\[ Q_k = \sum_{i=1}^{n} F_{i} \left( \frac{\partial x_i}{\partial q_k} \right) \]

\[ SW = \sum_{k=1}^{n} Q_k \delta q_k \]

Now work on 2nd term in \( \bigcirc \)

\[ \sum p_i \delta x_i = m \dot{x} \delta x_i = \sum_{i=1}^{n} m \dot{x} \left( \frac{\partial x_i}{\partial q_k} \right) \delta q_k \]

\[ \sum m \dot{x} \left( \frac{\partial x_i}{\partial q_k} \right) \delta q_k = \sum_{i=1}^{n} m, \left[ \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_k} \right) - \frac{\partial x_i}{\partial q_k} \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_k} \right) \right] \delta q_k \]
\[ \dot{x}_i = \dot{x}_i \quad \text{where} \quad \frac{\partial x_i}{\partial \dot{q}_0} = 0; \quad \frac{\partial x_i}{\partial q_0} \]

The partial derivatives \( \partial x_i / \partial q_0 \) keep all the other variables fixed except \( \dot{q}_0 \), so they can be functions of \( q_i \), \( q_{n-k} \), and time \( t \):

\[ \dot{x}_i = x_i \left( q_i, \ldots, q_{n-k}, \dot{q}_{n-k+1}, t \right) \]

From the derivative of

\[ \dot{x}_i = x_i \left( q_i, \ldots, q_{n-k}, t \right) \]

\[ \frac{\partial \dot{x}_i}{\partial q_0} = \frac{\partial x_i}{\partial q_0} \left( \text{all } q_i \text{ except } q_0 \text{ and } t \text{ are fixed} \right) \]

Look at:

\[ \sum m_i \frac{d}{dt} \left[ x_i \frac{\partial x_i}{\partial q_0} \right] = \sum m_i \frac{d}{dt} \left[ x_i \frac{\partial x_i}{\partial q_0} \right] \]

\[ = \frac{d}{dt} \frac{\partial}{\partial q_0} \left[ \frac{1}{2} \sum m_i x_i^2 \right] \]

Finally, rewrite:

\[ \frac{d}{dt} \frac{\partial x_i}{\partial q_0} = \frac{\partial}{\partial q_0} \left( \frac{d x_i}{dt} \right) \]

See text.
Now \( \sum \) becomes

\[
\sum_{i=1}^{n} m \dddot{x}_i = \frac{d}{dt} \frac{\partial}{\partial \dot{x}_i} \left( \frac{1}{2} \sum_{i} m \dot{x}_i^2 \right) - \sum_{i=1}^{n} m \dddot{x}_i \frac{d x_i}{dt}
\]

\[
\sum_{i=1}^{n} m \dddot{x}_i = \frac{d}{dt} \left( \frac{1}{2} \sum_{i} m \dot{x}_i^2 \right)
\]

Let \( T = \frac{1}{2} \sum m \dot{x}_i^2 \) \( \text{kinetic energy} \)

\[
\sum_{i=1}^{n} m \dddot{x}_i = \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_i} \right] - \frac{\partial T}{\partial x_i} \frac{d x_i}{dt}
\]

so \( \Box \) becomes

\[
\sum_{i=1}^{n} \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right) - \frac{\partial T}{\partial x_i} \frac{d x_i}{dt} \right] \frac{d q_i}{dt} = 0
\]