9/13/00

P521 Lecture 8

Go to colloquium today - B. Heckel !
Start reading Chapter 3 Lagrangian Dynamics

Reformulate Newton's Laws
a) Using only independent degrees of freedom
b) Incorporating constraints without complicated forces of constraint

Example: bead on a wire. Start with N Laws in 3 dim
i) 3 degrees of freedom $X(t)$, $Y(t)$, $Z(t)$
ii) Complex forces insure bead stays on wire

Alternatively only need one independent degree of freedom - for example displacement of bead along wire

Answer: Lagrangian Dynamics

$$L = T - V$$ note minus sign

Newton's Laws are equivalent to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here $\{q_\sigma\}$ are a set of generalized coordinates.

Constraints

Consider N particles with $n = 3N$ degrees of freedom

$$i = \underbrace{(1,2,3)}_{\vec{r}_1}\ \underbrace{(4,5,6)}_{\vec{r}_2} \cdots \underbrace{(n-2, n-1, n)}_{\vec{r}_N}$$

Holonomic Constraints

"Simple" explicit constraints that can be written as an equation

$$f_j(x_1, x_2 \ldots x_n; t) = 0$$

- assume we have k such constraints
- {For 1 bead on a wire k would $= 2$}

Example of nonholonomic constraint – particle constrained to move on or above sphere.

If particle had to stay on sphere – holonomic constraint.

Generalized coordinates

System with n degrees of freedom under k holonomic constraints has $n-k$ independent degrees of freedom

Let $q_1, q_2 \ldots q_{n-k}$ be any set of independent generalized coordinates that completely specify the configuration of system

$$x_1 = x_1(q_1 \ldots, q_{n-k}, t)$$

$$\vdots$$

$$x_n = x_n(q_1, \ldots, q_{n-k}, t)$$

Example for bead on wire $n=3$, $k=2$ and q_1 could be displacement of bead along wire. Shape of wire gives $x(q_1)$ $y(q_1)$, $z(q_1)$

Virtual Displacement

Infinitesimal, instantaneous displacement consistent with constraints.

Example small motion of bead along wire.

$$\delta x_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

Change in coordinates

change in generalized coordinates

From equations giving coordinates from generalized coordinates

D'alembert's principle

The reaction forces (forces of constraint) do no work under a virtual displacement

Example bead on a wire
Virtual displacement is along wire
Force of constraint is $\perp$ to wire to keep bead on wire
Force $\perp$ to displacement does no work

$$\dot{p}_i = F_i^a + R_i$$

Applied force

Reaction force

p_i = ith component of momentum

$$\sum_{i=1}^{n} (F_i^a + R_i - \dot{p}_i)\, \delta x_i = 0$$

given that $F_i^a + R_i - \dot{p}_i = 0$ for all i

D' principle $\sum_i R_i \, \delta x_i = 0$

or

$$\boxed{\sum_{i=1}^{n} (F_i^a - \dot{p}_i)\, \delta x_i = 0} \qquad *$$

Note all the δx_i are not in general independent. Need to change x, y, z in a correlated way to keep bead on wire.

Rewrite in terms of generalized coordinates where all the δq_i are independent.

Start with 1st term

$$\delta W \equiv \sum_i F_i \, \delta x_i$$

$$\delta W = \sum_{\sigma=1}^{n-k} \left[\sum_{i=1}^{n} F_i \left(\frac{\partial x_i}{\partial q_\sigma} \right) \right] \delta q_\sigma$$

Used $\delta x_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$

Generalized force

$$Q_\sigma \equiv \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma}$$

$$\boxed{\delta W = \sum_{\sigma=1}^{n-k} Q_\sigma \, \delta q_\sigma} \qquad *1$$

Now work on 2nd term in $*$

$$\sum \dot{p}_i \, \delta x_i = m_i \ddot{x}_i \, \delta x_i = \sum_\sigma \left(\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

$$\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} = \sum_i m_i \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma} \right) \right] \qquad *2$$

$$\dot{x}_i = \frac{d x_i}{d t} = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial x_i}{\partial t}$$

The partial derivatives $\frac{\partial x_i}{\partial q_\sigma}$ keep all the other variables fixed except q_σ so they can be functions of $q_1 \cdots q_{n-k}$ and time

$$\dot{x}_i = \dot{x}_i (q_1 \cdots q_{n-k} ; \dot{q}_1 \cdots \dot{q}_{n-k} ; t)$$

From time derivative of

$$x_i = x_i (q_1 \cdots q_{n-k} ; t)$$

$$\left. \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} \right| = \left. \frac{\partial x_i}{\partial q_\sigma} \right|$$ all q_i except q_σ and t are fixed

all variables $q_1 \cdots q_{n-k} ; \dot{q}_1 \cdots \dot{q}_{n-k}$ except $\dot{q}_\sigma ; t$) are kept fixed

Look at

$$\sum_i m_i \frac{d}{dt} \left[\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right] = \sum_i m_i \frac{d}{dt} \left[\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} \right]$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\sigma} \left[\frac{1}{2} \sum_i m_i \dot{x}_i^2 \right]$$

Finally rewrite

$$\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \left(\frac{d}{dt} x_i \right)$$

See text.

Now (* 2) becomes

$$\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\sigma} \left(\frac{1}{2} \sum_i m_i \dot{x}_i^2 \right) - \sum_i m_i \dot{x}_i \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt}$$

$$\sum_i m_i \dot{x}_i \frac{\partial}{\partial q_\sigma} \dot{x}_i = \frac{\partial}{\partial q_\sigma} \left(\frac{1}{2} m_i \dot{x}_i^2 \right)$$

Let $T \equiv \frac{1}{2} \sum_i m_i \dot{x}_i^2$ kinetic energy

$$\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_\sigma} \right] - \frac{\partial}{\partial q_\sigma} T$$

So (*) becomes

$$\sum_\sigma \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\sigma} \right) - \frac{\partial T}{\partial q_\sigma} - Q_\sigma \right] \delta q_\sigma = 0$$