9/6/00

Lecture 5 Cross Sections

General result

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

For inverse square law force

$$\cot\frac{\theta}{2} = \frac{V_\infty^2 b}{\gamma}$$

V_∞ = velocity at ∞
b = impact parameter

$$b = \frac{\gamma}{V_\infty^2}\frac{\cos\theta/2}{\sin\theta/2}$$

$$\frac{db}{d\theta} = -\frac{\gamma}{2V_\infty^2}\left[\frac{\sin\theta/2}{\sin\theta/2} + \frac{\cos^2\theta/2}{\sin^2\theta/2}\right]$$

$$= -\frac{\gamma}{2V_\infty^2}\frac{1}{\sin^2\theta/2}$$

$$\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \left|\frac{\gamma}{2V_\infty^2}\right|\frac{1}{\sin^2\theta/2}\quad\frac{\gamma}{V_\infty^2}\frac{\cos\theta/2}{\sin\theta/2}\quad\frac{1}{\sin\theta}$$

$$\sin\theta = 2\sin\theta/2\cos\theta/2$$

$$\boxed{\frac{d\sigma}{d\Omega}(\theta) = \left(\frac{\gamma}{2V_\infty^2}\right)^2\frac{1}{\sin^4\theta/2}}$$

Rutherford cross Section

Experimental Cross Section.

A detector of angular size $\Delta\Omega$ placed at an angle θ from a beam of flux F will count at a rate of

$$\text{Rate} = F \frac{d\sigma}{d\Omega}(\theta)\, \Delta\Omega$$

So

$$\frac{d\sigma}{d\Omega} = \frac{1}{F\,\Delta\Omega}\ \text{Rate}$$

This is simulated in program Sigma.bas. See web site ~~course~~ course lectures -> compute programs. To run program get QBASIC.EXE and type

C:> QBASIC Sigma

and then press alt R for run menu

Program generates N trajectories with impact parameters chosen at random in a square of side a

$$\text{Flux} = \frac{N}{a^2}$$

particles per unit time. Note Flux has dimensions of Length^{-2} time^{-1} so $\frac{d\sigma}{d\Omega}$ has dimensions of Length^2

Consider angular bins from $\theta_0 - \frac{\Delta\theta}{2} \to \theta_0 + \frac{\Delta\theta}{2}$

$$\Delta\Omega = 2\pi \sin\theta\, \Delta\theta$$

Total Cross Section for Asteroids to strike Earth

Consider all impact parameters for which

$$r_{min} \le R_E$$

$$r_{min} = (\epsilon - 1)\, a = \left(\frac{\epsilon - 1}{\epsilon + 1}\right)^{1/2} b$$

$$\sigma = 2\pi \int_0^{b_{max}} b\, db$$

where

$$r_{min}(b_{max}) = R_E \quad \text{radius of Earth}$$

$$R_E = r_{min} = \left[\frac{\left[1 + \left(\frac{V_\infty^2 b}{\gamma}\right)^2\right]^{1/2} - 1}{\left[1 + \left(\frac{V_\infty^2 b}{\gamma}\right)^2\right]^{1/2} + 1}\right]^{1/2} b$$

If V_∞^2 is large

$$R_E \simeq \left(\frac{V_\infty^2 b/\gamma - 1}{V_\infty^2 b/\gamma + 1}\right)^{1/2} b$$

$$\simeq \left(1 - \gamma / V_\infty^2 b\right) b$$

Correction term is small $R_E \simeq b$

$$b_{max} \simeq R_E\left(1 + \gamma / V_\infty^2 R_E\right)$$

$$\boxed{\sigma = \pi b_{max}^2 = \pi R_E^2 \left[1 + \frac{2GM_E}{V_\infty^2 R_E}\right]}$$

This is large than geometric cross section πR_E^2 because trajectories are focused by gravity. The extent of the focussing depends on mass of planet.

How to determine impact rate on Earth? Important for mass extinctions such as death of dinosaurs

Craters on Earth hard to find because of erosion.

① Count and date craters on moon

② $$\text{Rate}_{-\text{Earth}} = \text{Rate}_{-\text{moon}} \frac{\pi R_E^2 \left[1 + \frac{2GM_E}{V_\infty^2 R_E}\right]}{\pi R_m^2 \left[1 + \frac{2GM_m}{V_\infty^2 R_m}\right]}$$

since Earth and Moon probably exposed to same asteroid / comet flux

What is V_∞? Different for a comet and an asteroid

<u>For a comet</u>

Most Comets orbit the sun with very eccentric orbits $\epsilon \simeq 1$

$$\epsilon = \left(1 + \frac{2 E \ell^2}{m^3 \gamma^2}\right)^{1/2} \simeq 1$$

$$\Rightarrow \quad E \simeq 0$$

Energy of comet orbit is close to zero.

$$0 = V + T \Rightarrow T = -V$$

$$\tfrac{1}{2} m v^2 = \frac{G M_\odot m}{r}$$

$$v^2 = \frac{2 G M_\odot}{r_E}$$

This is velocity relative to Sun at point in orbit where $r = r_E$ = 1 astronomical unit Earth - Sun distance

Transfer this to velocity relative to Earth using known Earth orbit