Lecture 5 Cross Sections

General result
\[ \frac{d \sigma}{d \Omega} = \frac{b}{\sin \theta} \left| \frac{d b}{d \theta} \right| \]

For inverse square law force
\[ \cot \frac{\theta}{2} = \frac{V_{\infty} b}{\gamma} \quad V_{\infty} = \text{velocity at } \infty \]
\[ b = \frac{\gamma}{V_{\infty}} \cos \frac{\theta}{2} \]

\[ \frac{d b}{d \theta} = -\frac{\gamma}{2 V_{\infty}^2} \left[ \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} + \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \right] \]
\[ = -\frac{\gamma}{2 V_{\infty}^2} \frac{1}{\sin^2 \frac{\theta}{2}} \]
\[ \frac{b}{\sin \theta} \left| \frac{d b}{d \theta} \right| = \left| \frac{\gamma}{2 V_{\infty}^2} \right| \frac{1}{\sin^2 \frac{\theta}{2}} \frac{\gamma}{V_{\infty}^2} \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \frac{1}{\sin \theta} \]
\[ \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \]

\[ \frac{d \sigma(0)}{d \Omega} = \left( \frac{\gamma}{2 V_{\infty}^2} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \]

Rutherford cross section
Experimential Cross Section.

A detector of angular size $\Delta \Omega$ placed at an angle $\theta$ from a beam of flux $F$ will count at a rate of

$$\text{Rate} = F \frac{d\sigma(\theta)}{d\Omega} \Delta \Omega$$

So

$$\frac{d\sigma}{d\Omega} = \frac{1}{F \Delta \Omega} \text{ Rate}$$

This is simulated in program Sigma.bas. See web site. Course lectures \(\Rightarrow\) compute programs. To run program get QBASIC.EXE and type

:\> QBASIC Sigma

and then press alt R F6; run menu

Program generates N trajectories with impact parameters chosen at random

Flux = $\frac{N}{A}$

Particles per unit time. Note Flux

so $\frac{d\sigma}{d\Omega}$ has dimensions of Length$^{-2}$

Consider angular bins from $\theta_0 - \Delta \theta$ to $\theta_0 + \frac{\Delta \theta}{2}$

$\Delta \Omega = \frac{\pi}{2} \sin \theta \Delta \theta$
Total cross section for asteroids to strike Earth

Consider all impact parameters for \( \alpha \):

\[ r_{\min} \leq R_E \]

\[ r_{\min} = (\varepsilon - 1) \, \alpha = \left( \frac{\varepsilon - 1}{\varepsilon + 1} \right)^{1/2} b \]

\[ b_{\max} \]

\[ \sigma = 2\pi \int_0^{b_{\max}} b \, db \]

where \( r_{\min} (b_{\max}) = R_E \), radius of Earth

\[ R_E = r_{\min} = \left[ \frac{1 + \left( \frac{V_\infty^2 b}{V_\infty^2 b} \right)^{1/2}}{1 + \left( \frac{V_\infty^2 b}{V_\infty^2 b} \right)^{1/2} + 1} \right]^{1/2} b \]

If \( V_\infty^2 \) is large

\[ R_E = \left[ \frac{V_\infty^2 b^{1/2}}{V_\infty^2 b^{1/2} + 1} \right]^{1/2} b \]

\[ \approx (1 - \sqrt{1 - \frac{V_\infty^2 b}{V_\infty^2 b^{1/2} + 1}}) b \]

Correction term is small \( R_E \approx b \)

\[ b_{\max} = R_E \left( 1 + \sqrt{\frac{V_\infty^2 b}{V_\infty^2 b^{1/2} + 1}} \right) \]

\[ \sigma = \pi b_{\max}^2 = \pi R_E^2 \left[ 1 + \frac{2GM_E}{V_\infty^2 R_E} \right] \]
This is larger than geometric cross section by gravity. The extent of focusing depends on mass of planet.

How to determine impact rate on Earth?

Important for mass extinctions such as death of dinosaurs

Craters on Earth hard to find because of erosion.

6. Count and date craters or moon

\[
\text{Rate - Earth} = \text{Rate - moon} = \frac{\pi R_E^2 \left[ 1 + \frac{2GM_E}{V_{\infty}^2 R_E} \right]}{\pi R_m^2 \left[ 1 + \frac{2GM_m}{V_{\infty}^2 R_m} \right]}
\]

Since Earth and Moon probably exposed to same asteroid/comet flux

What is \( V_{\infty} \)? Different for a comet and an asteroid.

For a comet

Most comets orbit the Sun with very eccentric orbits \( \varepsilon = 1 \)

\[
\varepsilon = \left( \frac{2EL^2}{m^3\gamma^2} \right)^{1/2} = 1
\]

\( \Rightarrow \varepsilon = 0 \)

Energy of comet orbit is close to zero.
\[ 0 = V + T \implies T = -V \]

\[ \frac{1}{2} m v^2 = \frac{GM_0}{r} \]

\[ v^2 = 2 \frac{GM_0}{R_E} \]

This is velocity relative to Sun at a point in orbit, where \( R = R_E \) distance - 1 astronomical unit - Earth - Sun distance.

Transfer this to velocity relative to Earth using known Earth c/b.