9/4/06

P521 Lecture 4 Inverse Square Law Cont.

$$\phi = \phi_0 \mp \int^{u} du' \left[\frac{2mE}{l^2} + \frac{2m\gamma}{l^2} u' - u'^2\right]^{-1/2}$$

$\gamma = GM$, $u = 1/r$ $\quad l =$ ang. momentum

$$\frac{1}{r} = C\left[1 - \epsilon \cos(\phi - \phi_0)\right]$$

$$C = \frac{m^2\gamma}{l^2}, \quad \epsilon = \left[1 + \frac{2El^2}{m^3\gamma^2}\right]^{1/2}$$

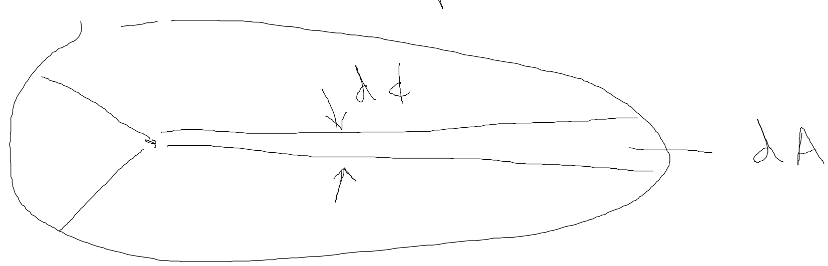
Semi-major axis of ellipse $a = \frac{1}{2}(r_{min} + r_{max})$

$$a = \frac{1}{C}\left(\frac{1}{1-\epsilon^2}\right) = -m\gamma/2E$$

$$\boxed{E = -\frac{m\gamma}{2a}}$$

Kepler's 2nd Law

Planets sweep equal areas in equal time



$dA = \frac{1}{2} r \, r \, d\phi$ $\quad$ ($\dot{\phi} = l/mr^2$ last time)

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\phi}{dt} = \frac{r^2}{2}\frac{l}{mr^2} = \frac{l}{2m} = \text{constant}$$

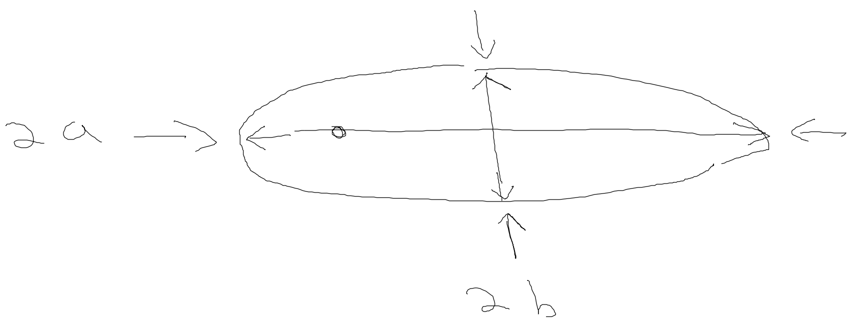
This follows just from angular momentum conservation. Independent of form of $V(r)$

Kepler's 3rd law

$$\int \frac{dA}{dt}\,dt = A = \pi a b = \frac{dA}{dt}\tau$$

πab = area of ellipse

b = semi minor axis $= a(1-\epsilon^2)^{1/2}$



$$A = \pi a^2 (1-\epsilon^2)^{1/2} = \frac{l}{2m}\tau$$

$$\boxed{\tau = \frac{2m}{l}\pi a^2 (1-\epsilon^2)^{1/2} = 2\pi a^{3/2}\gamma^{-1/2}}$$

Period of orbit depends only on $a^{3/2}$ and ~~is~~ is independent of ϵ.

Note $\epsilon = \left[1 + \frac{2El^2}{m^3\gamma^2}\right]^{1/2}$ so $\frac{(1-\epsilon^2)^{1/2}}{l} = \left(\frac{2E}{m^3\gamma^2}\right)^{1/2}$

Period is a way to measure mass of Sun since it depends on $\gamma = GM$

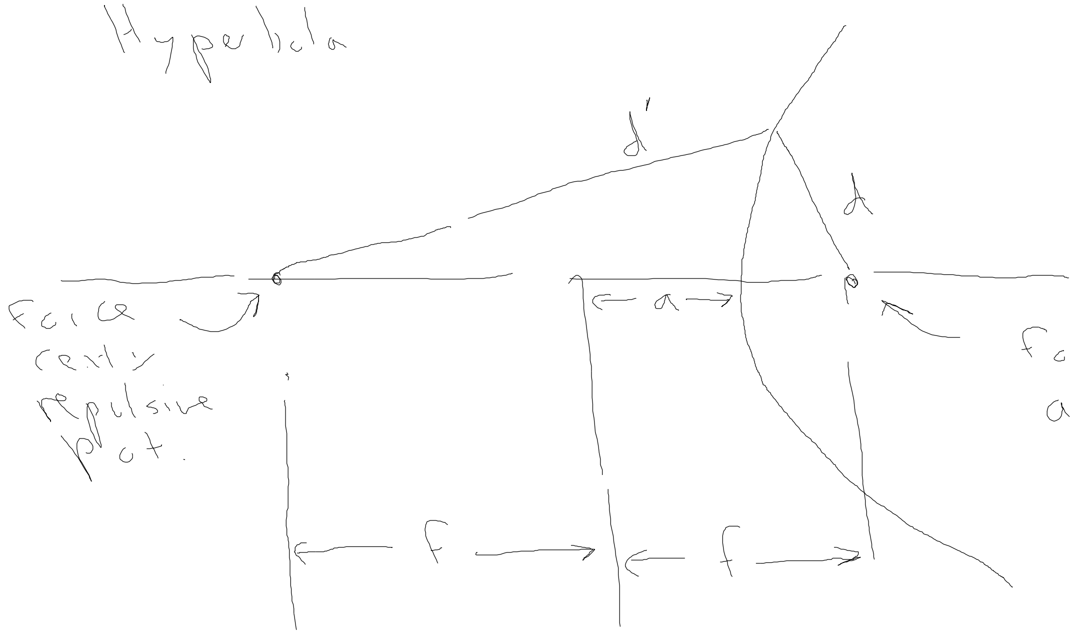
Scattering

$E < 0$, $\epsilon < 1$ bound ~~states~~ orbits

$E > 0$, $\epsilon > 1$ Scattering orbits, particle can escape to ∞

Hyperbolic Orbits in Gravitational Potential

Hyperbola



$d' - d = 2a$ def. of Hyperbola

$\epsilon = f/a > 1$

$$\epsilon = \left[1 + \frac{2El^2}{m^3\gamma^2}\right]^{1/2}$$

$E = \frac{1}{2} m v_\infty^2$ $\quad l = m v_\infty b$

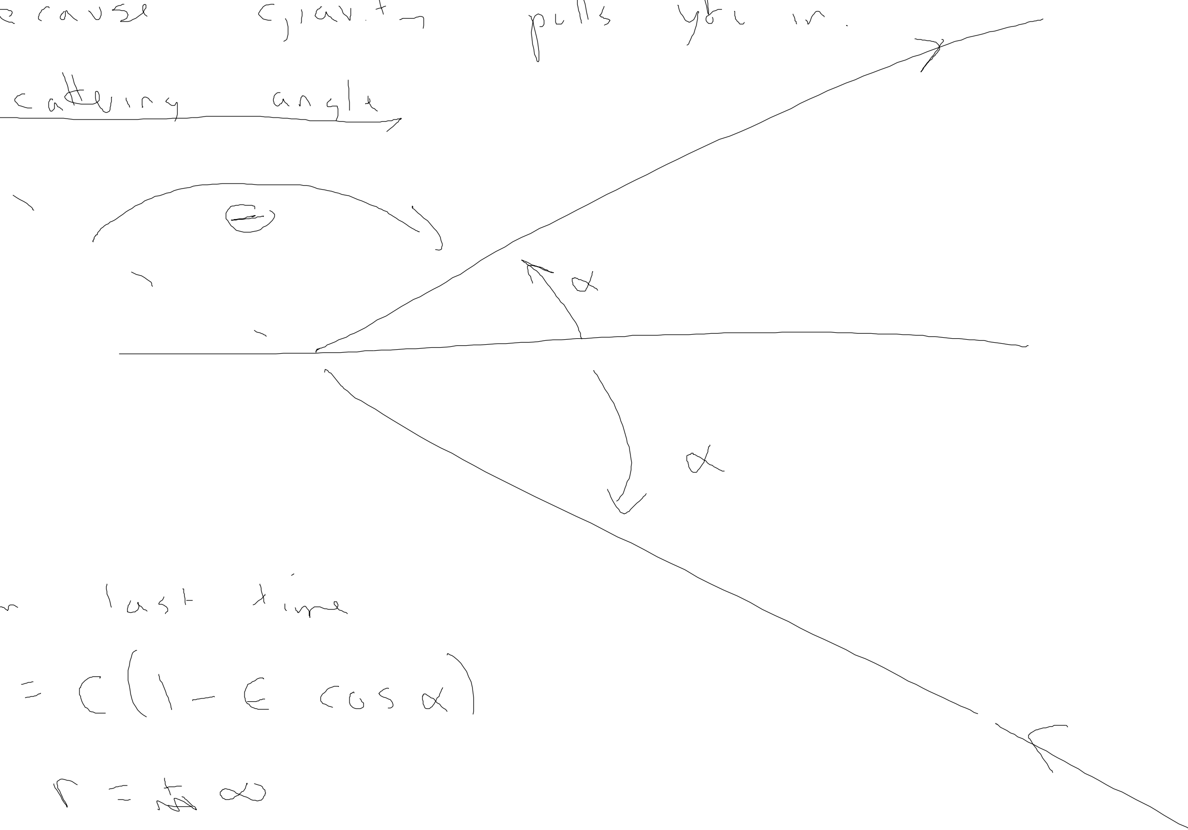
b = impact parameter (miss distance of straight line trajectory)

$$\epsilon = \left[1 + \left(\frac{v_\infty^2 b}{\gamma}\right)^2\right]^{1/2}$$

$$r_{min} = f - a = (\epsilon - 1)\, a = \left(\frac{\epsilon - 1}{\epsilon + 1}\right)^{1/2} b$$

distance of closest approach $r_{min} \le b$
because gravity pulls you in.

Scattering angle



From last time

$$\frac{1}{r} = C(1 - \epsilon \cos\alpha)$$

at $r = \pm\infty$

$$\alpha = \cos^{-1} \frac{1}{\epsilon}$$

Scatter angle $\Theta = \pi - 2\alpha$

$$\cot \tfrac{1}{2}\Theta = \cot\left(\frac{\pi}{2} - \alpha\right) = \tan\alpha = (\epsilon^2 - 1)^{1/2}$$

$$\boxed{\cot\left(\tfrac{1}{2}\Theta\right) = \frac{v_\infty^2\, b}{\gamma}}$$

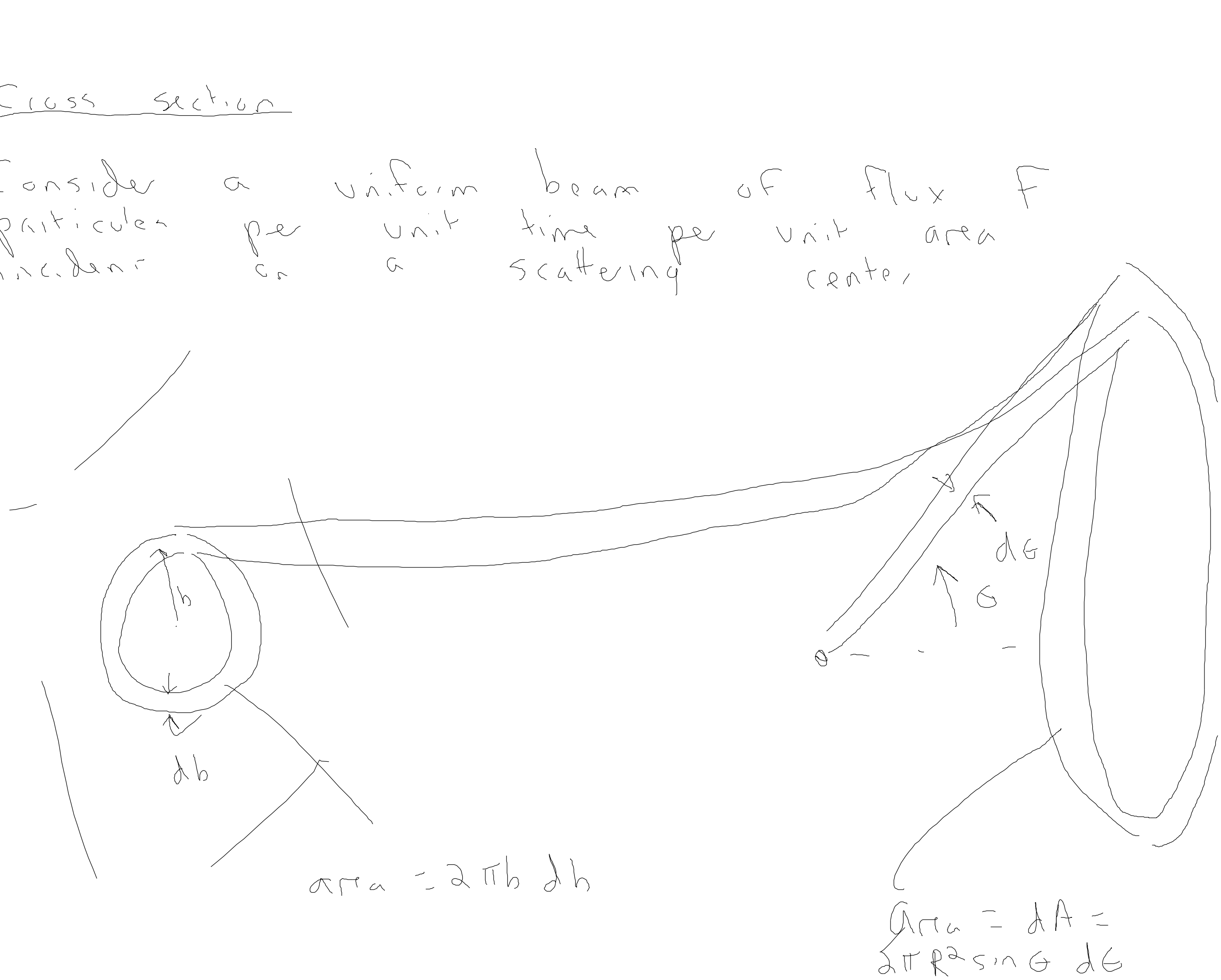
Gives scattering angle $\Theta = \Theta(b)$

Limits $b \to \infty$ $\Theta = 0$

$b \to 0$ $\Theta = \pi$

Cross section

Consider a uniform beam of flux F particles per unit time per unit area incident on a scattering center



$$F\, 2\pi b\, db = F\, d\sigma_{el}(\theta)$$

partial cross section

$$d\Omega = \frac{dA}{R^2} = 2\pi \sin\theta\, d\theta$$

solid angle

$$F\, d\sigma_{el} = F \left(\frac{d\sigma}{d\Omega}\right)_{el} d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

Calculate $\theta(b)$ and from it $\frac{db}{d\theta}$ for a general force law