P521 Lecture 3 Central Force Motion

Consider a particle of mass m moving in a fixed force field. Assume force directed along radius

$$\vec{F}(\vec{r}) \propto \vec{r}$$

$$\vec{F} = \hat{r}\, f(r) \qquad \vec{F} = -\vec{\nabla} V(\vec{r})$$

$$f = -\frac{dV(r)}{dr} \qquad \text{but } V \text{ is only a function of } r$$

$$\vec{r} \wedge \vec{F} = \text{torque} = 0$$

$$\vec{l} = \vec{r} \wedge \vec{p} = \vec{l}_0 \qquad \dot{l} \equiv 0$$

Orbital angular momentum is conserved with fixed magnitude and direction.

$\vec{l}$ defines a fixed direction in space and both $\vec{r}$ and $\vec{p}$ stay $\perp$ to this direction for all time

$$\vec{r} \cdot \vec{l} = \vec{p} \cdot \vec{l} = 0$$

$\Rightarrow$ Central force motion stays in a plane $\perp$ to $\vec{l}$. $\Rightarrow$ Only need 2 ~~dimensions~~ ~~and~~ coordinates instead of three.

Can chose coordinates so plane is defined by $z = 0$

$$x = r\cos\phi \qquad y = r\sin\phi$$

<u>Important</u> We will often want to calculate kinetic energy of a particle in some new coordinates (example r, ϕ)

Always start with cartes. coordinates

(1) $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

(2) Calculate $\dot{x}, \dot{y}, \dot{z}$ in terms of your new coordinates (r, ϕ) and their time derivatives $\dot{r}, \dot{\phi}$

$\dot{z} = 0$ $\quad x = r\cos\phi$ $\quad y = r\sin\phi$

$\dot{x} = \dot{r}\cos\phi + r\sin\phi\,\dot{\phi}$ $\quad \dot{y} = \dot{r}\sin\phi - r\cos\phi\,\dot{\phi}$

$\dot{x}^2 + \dot{y}^2 = r^2(\sin^2\phi + \cos^2\phi)\dot{\phi}^2 + \dot{r}^2$

(3) $\boxed{T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2)}$

Conservation of energy

$E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$

Called first integral of motion since it only involves 1st time derivative. [Newton's 2nd law involves 2nd time derivative.]

Conservation of angular momentum

$\vec{l} = r \wedge p \Rightarrow l_z = x p_y - y p_x$

$\vec{l} = \hat{z} l_z \rightarrow (x\dot{y} - y\dot{x})\, m = \boxed{m r^2 \dot{\phi} = l_z}$ $\quad l_x = l_y = 0$

$l_z = l_0 =$ constant

$\boxed{\dot{\phi} = l_0 / m r^2}$

$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \overbrace{\left(\frac{l_0^2}{m^2 r^4}\right)}^{\dot{\phi}^2} + V(r)$

Define

$$\boxed{V_{eff} = \frac{l_c^2}{2mr^2} + V(r)}$$

Equivalent one dimensional problem

$$E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r)$$

or

$$\dot{r} = \pm \left[\frac{2}{m}\left(E - V_{eff}(r)\right)\right]^{1/2}$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} \qquad \dot{\phi} = l_c/mr^2$$

$$\frac{dr}{d\phi} = \left(\frac{d\phi}{dt}\right)^{-1}\left[\frac{2}{m}\left(E - V_{eff}\right)\right]^{1/2}$$

$$\frac{d\phi}{dr} = \pm \frac{l_c}{(2m)^{1/2}} \frac{1}{r^2} \frac{1}{(E - V_{eff})^{1/2}}$$

Integrate above equation

$$\boxed{\phi = \phi_0 \pm \frac{l_c}{(2m)^{1/2}} \int^r \frac{dr'}{r'^2} \frac{1}{\left[E - V_{eff}(r')\right]^{1/2}}}$$

Note this is a very compact expression which depends on two constants of the motion l_c and E

To evaluate integral need form of $V(r)$
For general V need to calculate integral numerically

Inverse Square Force $\rightarrow$ Kepler's Laws

$V(r) = - m\gamma / r$ $\qquad \boxed{\gamma \equiv GM}$

let $\boxed{u = \frac{1}{r}}$

$V_{eff} = - m\gamma u + \frac{l_c^2}{2m} u^2$ quadratic expression

$$\phi = \phi_0 \mp \int^{u} du' \left[\frac{2mE}{l_c^2} + \frac{2m\gamma u'}{l_c^2} - u'^2 \right]^{-1/2}$$

This is a standard integral

$$\phi = \phi_0 \pm \cos^{-1}\left[\frac{1 - u\, l_c^2/m^2\gamma}{(1 + 2E\, l_c^2/m^3\gamma^2)^{1/2}} \right]$$

solve for $u = 1/r$

$$u = \frac{1}{r} = C\left[1 - \epsilon \cos(\phi - \phi_0)\right]$$

$$C = \frac{m^2\gamma}{l_c^2} \qquad \epsilon = \left[1 + \frac{2E/m}{m^2\gamma^2/l_c^2}\right]^{1/2}$$

Expect motion to be independent of mass of particle $\Rightarrow$

l_c/m = angular momentum per unit mass

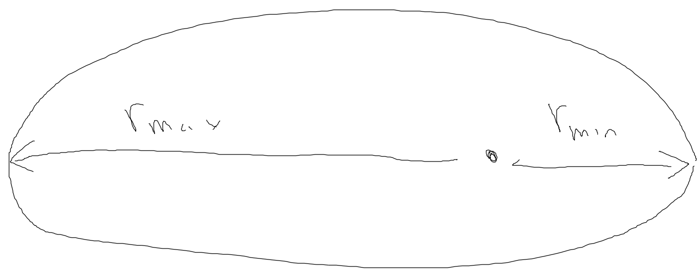
E/m = Energy per mass

Otherwise no other dependence on mass m

$$\frac{1}{r} = C\left[1 - \epsilon \cos(\phi - \phi_0)\right]$$

Is the equation of an ellipse

$$r_{min} = \frac{1}{C}\frac{1}{1+\epsilon} \qquad r_{max} = \frac{1}{C}\frac{1}{1-\epsilon}$$



Semimajor axis $\quad a = \frac{1}{2}(r_{min} + r_{max})$

$$a = \frac{1}{C}\frac{1}{1-\epsilon^2} = -\frac{m\gamma}{2E}$$

or

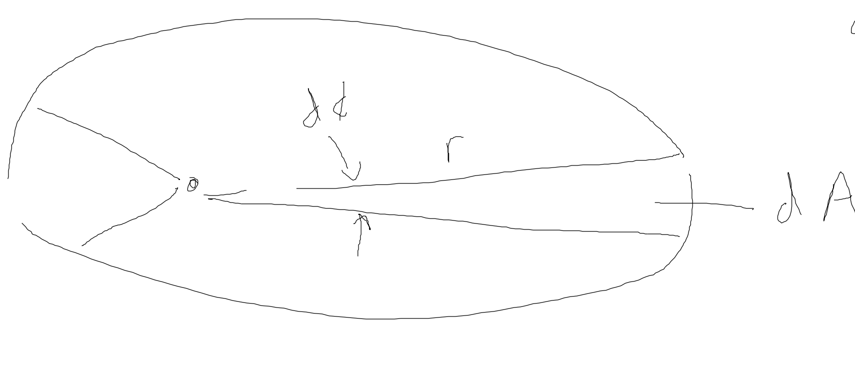
$$\boxed{E = -\frac{m\gamma}{2a} = -\frac{GMm}{2a}}$$

Energy of ellipse depends only on semimajor axis and is independent of ϵ

<u>Kepler's 2nd Law</u>

Planets sweep equal areas in equal time

$$dA = \frac{1}{2} r \, r d\phi$$



$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\phi}{dt} = \frac{r^2}{2}\frac{l_c}{mr^2} = \frac{l_c}{2m} = \text{constant}$$

independent of r

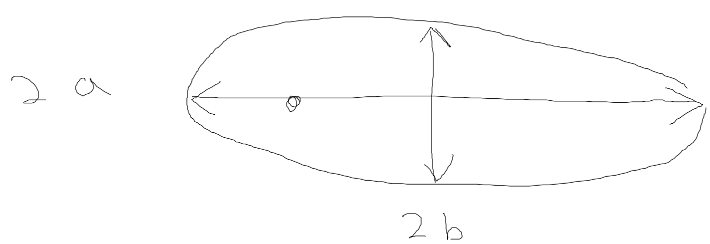
Kepler's 2nd law follows from angular momentum conservation independent of radial form of $V(r)$

Kepler's 1st law ellipses of course needs $1/r^2$ force (or r^2 force with ellipse centered on origin)

Period of orbit and Kepler's 3rd law

$$\int \frac{dA}{dt}\,dt = A = \pi ab \quad \text{area of ellipse}$$

$$b = a(1-\epsilon^2)^{1/2} \quad \text{semiminor axis}$$



$$A = \pi a^2(1-\epsilon^2)^{1/2} = \frac{dA}{dt}\tau = \frac{l_c}{2m}\tau$$

solve for period τ

$$\boxed{\tau = \frac{2m}{l_c}\pi a^2(1-\epsilon^2)^{1/2} = 2\pi a^{3/2}\gamma^{-1/2}}$$