10/30/00

Lecture 29, The intra-Mercurial planet Vulcan

Discuss precession of Mercury's perihelion

Oscillates in radius with period $T_r$
Goes around in $\phi$ with period $T_\phi$

Precession of perihelion

$$\Delta\phi = 2\pi \left[ \frac{T_r - T_\phi}{T_\phi} \right]$$ radians per orbit

Let $V = -\frac{\gamma m}{r} + \delta V$ $\qquad \gamma \equiv G M_\odot$

$$T_r = 2\pi \left\{ m \Big/ \left[ \frac{3V'}{r} + V'' \right] \right\}^{1/2}$$

$$\frac{V'}{r} = \frac{\gamma m}{r^3} + \frac{\delta V'}{r}, \qquad V'' = -\frac{2\gamma m}{r^3} + \delta V''$$

$$T_r = 2\pi \left[ m \Big/ \left( \frac{\gamma m}{r^3} + \frac{3\delta V'}{r} + \delta V'' \right) \right]^{1/2}$$

$$T_r = 2\pi \left( \frac{r^3}{\gamma} \right)^{1/2} \left[ 1 - \frac{r^3}{2\gamma m} \left( \frac{3\delta V'}{r} + \delta V'' \right) \right]$$

$$T_\phi = 2\pi \left[ \frac{mr}{V'} \right]^{1/2} = 2\pi \left( \frac{r^3}{\gamma} \right)^{1/2} \left( 1 - \frac{r^3}{2\gamma m} \frac{\delta V'}{r} \right)$$

$$\boxed{\Delta\phi = -\pi \frac{r}{\gamma} \frac{2\delta V' r + r^2 \delta V''}{m}}$$

In General relativity $\Phi = -\frac{\gamma}{r} \left[ \frac{1}{1 - \frac{2GM}{rc^2}} \right]$

$$\Rightarrow \delta V = -\alpha \frac{\gamma m}{r} \left( \frac{\gamma}{rc^2} \right) \qquad \alpha = 3$$

$\frac{2GM_\odot}{c^2}$ = Schwartzchild radius of black hole

For sun $\frac{GM_\odot}{c^2} = \frac{\gamma}{c^2} = 1.475$ km

$\delta V' = 2\alpha \frac{\gamma^2 m}{r^3 c^2}$ $\qquad \delta V'' = -6\alpha \frac{\gamma^2 m}{r^4 c^2}$

$$\boxed{\Delta\phi = 2\pi\alpha\left(\frac{\gamma}{rc^2}\right)}$$

$= 2\pi(3) \frac{1.475\text{ km}}{5.79\times10^7\text{ km}} = 4.8\times10^{-7}\ \frac{\text{radians}}{\text{orbit}}$

r = radius of Mercury's orbit $= (.241)^{2/3} a_0$
$T_0$ = 0.241 years

$\Delta\phi = \left[\frac{100\text{ Y / cent}}{0.241\text{ Y / orbit}}\right]\left[\frac{180^\circ}{\pi\text{ radians}}\right]\left[3600\ \frac{\text{sec of arc}}{\text{deg.}}\right]$
$4.8\times10^{-7}$ radians / orbit

$$\boxed{\Delta\phi = 41''\text{ of arc / century}}$$

Assume nearly circular orbit
Correction factor of $\frac{1}{1-\epsilon^2}$ for elliptical orbit

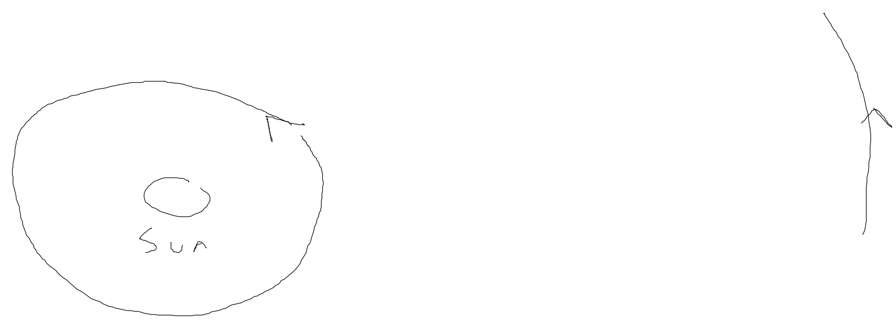
$\Delta\phi = 2\pi\alpha \frac{\gamma}{a(1-\epsilon^2)c^2} = 43''$ / century

$\epsilon = .206$

Alternative explination for precession

Planet Vulcan was postulated to be another planet inside orbit of Mercury

It would orbit rey quickly so time average its position into a ring of mass




Mercury's orbit

Vulcan's orbit

Multipole expansion of Vulcan's gravitational field

$$\Phi_v \approx -\frac{G m_v}{r} - \frac{G q_2}{r^3} P_2(\cos\theta) + \dots$$

$$q_2 = \int d^3r' \, \rho(r') \, r'^2 \, P_2(\cos\theta')$$

$\theta' = 90^\circ$ $\quad$ $P_2(90^\circ) = -\frac{1}{2}$

$$q_2 = -\frac{1}{2} m_v r_v^2$$

In same plane $\theta \approx 90^\circ$ $\quad$ $P_2(\cos\theta) \approx -\frac{1}{2}$

$$\Phi_v = -\frac{G m_v}{r} - \frac{G m_v r_v^2}{4 r^3} + \dots$$

$$\boxed{\delta V = -G m_v m_m \frac{r_v^2}{4 r^3}}$$

interaction between Vulcan and Mercury

$$\Delta\phi = -\left(\frac{\pi r}{\gamma}\right)\left[2r \quad 3\frac{G m_v r_v^2}{4 r^4} - \frac{12 G m_v r_v^2}{4 r^3}\right]$$

$$\Delta\phi = \frac{3}{2}\pi\left(\frac{G m_v}{\gamma}\right)\frac{r_v^2}{r^2}$$

$$\boxed{\Delta\phi = \frac{3\pi}{2}\left(\frac{m_v}{M_\odot}\right)\frac{r_v^2}{r^2}} \qquad = 4.8 \times 10^{-7} \ \frac{\text{rad.}}{\text{orbit}}$$

observed

$$m_v \approx \left(\frac{r^2}{r_v^2}\right) 1.0 \times 10^{-7} \ M_\odot$$

If $r_v = \frac{1}{2} r$ say

$$\boxed{m_v = 4 \times 10^{-7} \ M_\odot = 0.14 \ M_{\text{Earth}}}$$

Or string of asteroids
Or quadrupole moment of Sun

Note searched for Vulcan as it would transit Sun.

$\boxed{\text{Start Reading Chap. 6}}$