10/20/06

Lecture 24 Precession of Equinoxes

Gravitational pot. of Earth from Sun

$$V = -\frac{GM_e M_\odot}{r} + \frac{GM_e M_\odot}{r^3} J_2 R_e^2 \, P_2(\cos\theta) + \ldots$$

$$J_2 = \frac{I_3 - I_1}{M_e R_e^2}$$

See lecture 20 notes Oct 11

Assume Sun is located in $\hat{e}_2^0$ direction

$$\cos\theta = \hat{e}_3 \cdot \hat{e}_2^0 = \sin\beta \sin\alpha$$

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2} = \frac{3}{2}\sin^2\beta \sin^2\alpha + \text{const.}$$

Drop constant term (including monopole $1/r$ term)

$$V = \frac{3}{2}\frac{GM_\odot}{r^3}(I_3 - I_1)\sin^2\beta \sin^2\alpha$$

$$\boxed{V = -\lambda \sin^2\beta \sin^2\alpha} \qquad \lambda \equiv -\frac{3}{2}\frac{GM_\odot}{r^3}(I_3 - I_1)$$

Earth is $\simeq$ symmetric top $I_1 \simeq I_2$

$$L = T - V$$

$$\boxed{L = \tfrac{1}{2} I_1 (\dot\alpha^2 \sin^2\beta + \dot\beta^2) + \tfrac{1}{2} I_3 (\dot\alpha \cos\beta + \dot\gamma)^2 + \lambda \sin^2\alpha \sin^2\beta}$$

$\dot\gamma$ is rotation about e_3 $\longrightarrow$ $I_3 \dot\gamma^2/2 = T$

$\dot\beta$ is rotation axis in equitorial plane I_1 or $I_2 \simeq I_1$ $\longrightarrow$ $I_1 \dot\beta^2$

$\dot\alpha$ has projection $\cos\beta$ on $\hat{e}_3$ and $\sin\beta$ in equitoral plane $I_1 \dot\alpha^2 \sin^2\beta + I_3 \dot\alpha^2 \cos^2\beta$

Lagrange Equations

$\boxed{\beta}$ $\quad \dfrac{\partial L}{\partial \dot\beta} = I_1 \dot\beta \qquad \dfrac{\partial L}{\partial \beta} = I_1 \dot\alpha^2 \sin\beta\cos\beta - I_3(\dot\alpha\cos\beta+\dot\gamma)\,\dot\alpha\sin\beta + 2\lambda \sin\beta\cos\beta\sin^2\alpha$

$$\boxed{I_1\ddot\beta - I_1\dot\alpha^2\sin\beta\cos\beta - I_3(\dot\alpha\cos\beta+\dot\gamma)\,\dot\alpha\sin\beta - 2\lambda\sin\beta\cos\beta\sin^2\alpha = 0}$$

$\boxed{\alpha}$ $\quad \dfrac{\partial L}{\partial \dot\alpha} = I_1\dot\alpha\sin^2\beta + I_3(\dot\alpha\cos\beta+\dot\gamma)\cos\beta$

$$\frac{\partial L}{\partial \alpha} = 2\lambda\sin^2\beta\sin\alpha\cos\alpha$$

$$\boxed{\frac{d}{dt}\left[I_1\dot\alpha\sin^2\beta + I_3(\dot\alpha\cos\beta+\dot\gamma)\cos\beta\right] - 2\lambda\sin^2\beta\cos\alpha\sin\alpha = 0}$$

$\boxed{\gamma}$ $\quad \dfrac{\partial L}{\partial \dot\gamma} = I_3(\dot\alpha\cos\beta+\dot\gamma) \qquad \dfrac{\partial L}{\partial \gamma} = 0$

γ is cyclic $\qquad p_\gamma \equiv \dfrac{\partial L}{\partial\dot\gamma} = I_3(\dot\alpha\cos\beta+\dot\gamma)$

$$\frac{d}{dt}p_\gamma = 0 \Rightarrow p_\gamma = \boxed{I_3(\dot\alpha\cos\beta+\dot\gamma) = \text{const}}$$

<u>$\lambda = 0$ solutions</u>

$\dot\alpha = \dot\beta = \ddot\beta = 0$

$\beta = \beta_0 = 23.5^\circ$ Obliquity of Earth

$p_\gamma = I_3\dot\gamma$

$$p_\alpha = p_\gamma\cos\beta_0 \equiv I_1\dot\alpha\sin^2\beta + p_\gamma\cos\beta = \frac{\partial L}{\partial\dot\alpha}$$

Expand to first order in $\dot\alpha$, $\dot\beta$, λ and

$$\beta \simeq \beta_0 + \eta_\beta$$

Time average position of sun

$$\langle \sin^2\alpha \rangle = \frac{1}{2}$$

$$\langle \sin\alpha \cos\alpha \rangle = 0$$

$\boxed{\alpha}$

$$\frac{d}{dt}\left[p_\alpha\right] = 2\lambda \sin^2\beta_0 \langle \sin\alpha \cos\alpha \rangle$$

$$\simeq 0$$

$$p_\alpha = \text{const}$$

$$p_\alpha = I_1 \dot\alpha \sin^2\beta + p_\gamma \cos\beta$$

$\dot\alpha$ is first order so $\beta \simeq \beta_0$ in first term

$$\cos\beta \simeq \cos\beta_0 - \sin\beta_0 \, \eta_\beta$$

$$p_\alpha \simeq I_1 \dot\alpha \sin^2\beta_0 + p_\gamma \cos\beta_0 - p_\gamma \sin\beta_0 \, \eta_\beta$$

But from $\lambda = 0$ solution $\quad p_\alpha = p_\gamma \cos\beta_0$

$$\boxed{\eta_\beta = \frac{I_1 \sin\beta_0 \, \dot\alpha}{p_\gamma}}$$

$\boxed{\beta}$ Drop $\dot\alpha^2$ term in β equation

$$I_1 \ddot\eta_\beta - p_\gamma \dot\alpha \sin\beta_0 = \lambda \sin\beta_0 \cos\beta_0$$

$$I_1 \ddot\eta_\beta = -\omega_p^2 I_1 \eta_\beta = -\omega_p^2 \left[\frac{I_1^2 \sin\beta_0}{p_\gamma}\right] \dot\alpha$$

$$\ll p_\gamma \dot\alpha \sin\beta_0$$

by of wide ω_p^2 / ω^2

where $\omega = \dot{\gamma}$ = rotation rate of Earth

So drop $\ddot{\eta}_\beta$ term

$$\boxed{\dot{\alpha} \simeq - \frac{\lambda \cos\beta_0}{P_\gamma}}$$

$\dot{\alpha}$ is rate of Precession of Equinoxes

$P_\gamma = I_3 \omega$

$$\boxed{\dot{\alpha} = \frac{3}{2} \frac{(I_3 - I_1) G}{I_3 \omega} \left[\frac{M_\odot}{r_{se}^3} \cos\beta_0 + \frac{M_m}{r_{me}^3} \langle \cos\beta_m \rangle \right]}$$

added the contribution from the Moon

[Goes like $\frac{M}{r^3}$] Moon orbit is approx. in plane

$\langle \cos\beta_m \rangle \simeq \cos\beta_0$

approx. Neglected eccentricity and that r is time dependent (small correction)

Numbers

$\omega = 7.29 \times 10^{-5}$

$r_{se} = 1.496 \times 10^{13}$ cm

$M_\odot = 1.989 \times 10^{33}$ g

$G = 6.67 \times 10^{-8}$ (cgs)

$M_m = 7.4 \times 10^{25}$ g

$r_{em} = 3.9 \times 10^{10}$ cm

$\frac{I_3 - I_1}{I_3} = \frac{1}{305.3}$

$\beta_m \simeq \beta_0 = 23.5^\circ$

$$\dot{\alpha} = \frac{3}{2}\left[\frac{6.67\times10^{-8}}{7.29\times10^{-5}}\right]\left[\frac{1}{305.3}\right]\left[\frac{1.99\times10^{33}}{(1.56\times10^{13})^3} + \frac{7.4\times10^{22}}{(3.9\times10^{10})^3}\right]\cos 23.5^\circ$$

$$\dot{\alpha} \simeq 7.6\times10^{-12}\ s^{-1}$$

Period $\boxed{T = \frac{2\pi}{\dot{\alpha}} = 26000 \text{ years}}$

Notes

(1) $(I_3 - I_1)/I_3$ determined from measured $\dot{\alpha}$
However can estimate $(I_3 - I_1)/I_3$ from measured R_E and R_{pole} of Earth
Or just set

$$I_3 \simeq 0.4\ M_e R_e^2$$

and close to real value $I_3 \simeq 0.33\ M_e R_e^2$

So can get a good estimate of 26000 years. First done by Newton.

(2) Precession rate for Mars

a) No big Moon! $\dot{\alpha}$ for Earth increased by ~ 3 because of Moon
b) $r = 1.52$ AU and $\dot{\alpha} \propto 1/r^3$
c) day almost exactly the same as earth
d) $J_2 = \frac{I_3 - I_1}{MR^2} = 0.001960$ Mars

0.001083 Earth

e) I_3 not exactly known for Mars
Best models $I_3 \simeq 0.365\ MR^2$
compared to $I_3 = 0.33\ MR^2$ for earth.

$(I_3 - I_1)/I_3 \simeq 1/186$

f) $\beta_0 \simeq 25.2^\circ$ very close to Earth

$\dot{\alpha} \approx \mp 1.2 \times 10^{-12}\ s^{-1}$ $\boxed{P \approx 170{,}000 \text{ Years}}$

For an elliptical orbit or for perturbations from another planet

$$\left\langle \frac{\sin\alpha \cos\alpha}{r^3} \right\rangle$$

may not average to zero.

Look at equation for P_α

$$\frac{d}{dt}\left[P_\alpha\right] = \frac{d}{dt}\left[\frac{I_1 \lambda \cos\beta \sin^2\beta}{P_\gamma} + P_\gamma \cos\beta\right] = 2\lambda \sin^2\beta \langle \sin\alpha \cos\alpha \rangle$$

Used $P_\alpha = I_1 \dot{\alpha} \sin^2\beta + P_\gamma \cos\beta$ and $\dot{\alpha} = \dfrac{-\lambda \cos\beta}{P_\gamma}$

Work to lowest order in λ or β

$$\frac{d}{dt}\left[I_1 \frac{\lambda \cos\beta_0 \sin^2\beta_0}{P_\gamma} + P_\gamma \cos\beta\right] = 2\sin^2\beta_0 \langle \lambda \sin\alpha \cos\alpha \rangle$$

$$\boxed{\frac{d}{dt} \cos\beta = \frac{2\sin^2\beta_0}{P_\gamma} \langle \lambda \sin\alpha \cos\alpha \rangle}$$

If time average is nonzero Obliquity will change with time $\rightarrow$ Large effects on climate.