10/18

Lecture 23 Hyperion and Chaos

Last time Triaxial body can undergo stable rotation about largest or smallest moment of inertia. Rotation about 2nd axis is unstable $\Rightarrow$ small amplitude motions about other two axes will grow with time

Hyperion is an irregular moon of Saturn 410 x 260 x 220 km in size. It rotates chaoticly.

Unstable vs Chaotic

1. Start with small amplitude osc. analysis

2. Check for stability: all $\omega^2 > 0$?

3. Stable $\Rightarrow$ Expect regular motion. Small perturbations should only have small effects. One exception: perturbation applied exactly at a normal mode frequency is in resonance and can lead to large amplitude motion. Example Kirkwood gaps in asteroid belt.

4. Unstable $\Rightarrow$ implies large amplitude motion. Need to go back and look at full nonlinear equations of motion.

5. Large amplitude motion can be regular or chaotic.

6. Chaotic motion $\Leftrightarrow$ extreme sensitivity to inital conditions.

Example, integrate solar system in time and calculate position of Pluto 1 million years in future. Now move Pluto's present position slightly and repeat the integration. Compare the two future positions of Pluto both 1 million years from now.

Regular motion: Future positions close together if inital conditions close.

Chaotic motion: Small changes in inital conditions can lead to large changes in future positions.

Short term predictions are fine, even for Chaotic motion. However long term predictions of chaotic motion is very hard because of accuracy ~~with wh~~ needed for inital conditions and sensitivity to small perturbations.

Example: In 1960s a simplified model of weather was shown to be chaotic. This is a property of nonlinear hydrodynamics equations which govern weather ⟶ Long term weather forcasting is very hard.

Popular book on Chaos in solar system

"Newton's Clock" by I. Peterson, Freemant Co, NY.

Moon rotates stabily about its largest moment. Tidal dissipation has locked rotation so that heavier near side always faces Earth.

Amalthea rotates stably about its smallest moment with its long axis pointing to Jupiter.

An irregular moon can start out with unstable rotation about middle axes. However expect tidal dissipation to quickly lock spin into stable rotation about 1st or 3rd axis.

This did not happen with Hyperion because Titan keeps kicking it. Hyperion is in a 4:3 resonance with Titan. Titan goes around 4 times for every 3 orbits of Hyperion.

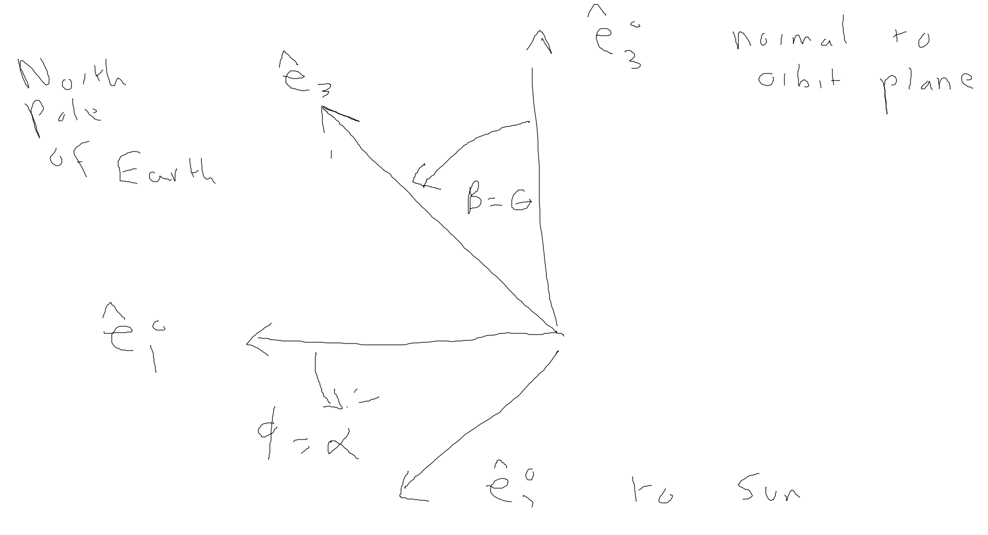
Euler Angles (α, β, γ)

Efficient way of both specifying the orientation of a body w.r.t. some inertial frame and to describe its rotational velocity

Consider inertial frame fixed w.r.t. Sun.

$\hat{e}_3^{\circ}$ is normal to plane of Earth's orbit.

$\hat{e}_1^{\circ}, \hat{e}_2^{\circ}$ define plane of Earth's orbit
Let $\hat{e}_2^{\circ}$ point from Earth to Sun



Earth sits at common origin of $\hat{e}_i^0$ (inertial) and $\hat{e}_i$ (body fixed frame)

$\hat{e}_3$ points to North pole

$\hat{e}_1$ and $\hat{e}_2$ are in plane of equator
[$\hat{e}_1$ is at $0°$ latitude say]

Let polar angles $\Theta = \beta$ and $\phi = \alpha$ specify orientation of North pole

$\beta = 23.5°$ Earth's obliquity very important for climate

Need one more angle γ to specify orientation of Earth about $\hat{e}_3$ axes.

$\dot{\gamma} = \omega \qquad 1 \text{ day} \sim \frac{2\pi}{\dot{\gamma}}$

Euler angles are defined as a transformation from $\hat{e}_i^0$ to $\hat{e}_i$

1. Rotate about $\hat{e}_3^0$ by α to bring $\hat{e}_2$ into direction of "line of nodes"

"Line of nodes" intersection of orbit plane with plane of equator

2. Rotate about line of nodes by β bringing $\hat{e}_3$ into final orientation

3. Rotate about new $\hat{e}_3$ by γ.

This completely determines orientation of body.

$$\hat{e}_\gamma = \hat{e}_3$$

$$\hat{e}_\beta = \hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma$$

$$\hat{e}_\alpha = \hat{e}_3 \cos\beta + \hat{e}_2 \sin\beta \sin\gamma - \hat{e}_1 \sin\beta \cos\gamma$$

$$\vec{\omega} = \dot{\alpha}\,\hat{e}_\alpha + \dot{\beta}\,\hat{e}_\beta + \dot{\gamma}\,\hat{e}_\gamma$$

Note $\hat{e}_\alpha$, $\hat{e}_\beta$, $\hat{e}_\gamma$ are not ortogonal but still add as vectors

$$\omega_1 = \hat{e}_1 \cdot \vec{\omega} = -\dot{\alpha} \sin\beta \cos\gamma + \dot{\beta} \sin\gamma$$

$$\omega_2 = \hat{e}_2 \cdot \vec{\omega} = \dot{\alpha} \sin\beta \sin\gamma + \dot{\beta} \cos\gamma$$

$$\omega_3 = \dot{\alpha} \cos\beta + \dot{\gamma}$$

$$T = \tfrac{1}{2}\left[I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right]$$

General expression is a mess.

Consider symmetric top like Earth with $I_1 = I_2$

$$T = \tfrac{1}{2}\left[I_1 (\omega_1^2 + \omega_2^2) + I_3 \omega_3^2\right]$$

$$\boxed{T = \tfrac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \tfrac{1}{2} I_3 (\dot{\alpha} \cos\beta + \dot{\gamma})^2}$$