Lecture 23: Hyperion and Chaos

Last time: Triaxial body can undergo stable rotation about largest or smallest moment of inertia. Rotation about 2nd axis is unstable ⇒ small amplitude motions about other two axes will grow with time.

Hyperion is an irregular moon of Saturn, 440 × 260 × 220 km in size. It rotates chaotically.

Unstable vs Chaotic

1. Start with small amplitude o-sc. analysis.

2. Check for stability: all \( w^2 > 0 \) ?

3. Stable ⇒ expect regular motion. Small perturbations should only have small effects. One exception: perturbation applied exactly at a normal mode frequency is in resonance and can lead to large amplitude motion. Example: Kirkwood gaps in asteroid belt.

4. Unstable ⇒ implies large amplitude motion. Need to go back and look at full nonlinear equations of motion.

5. Large amplitude motion can be regular or chaotic.

6. Chaotic motion ⇒ extreme sensitivity to initial conditions. Example: integrate solar system in time and...
Calculate position of Pluto 1 million years in future. Now move Pluto's present position slightly and repeat the integration. Compare the two future positions of Pluto both 1 million years from now.

Regular motion: Future positions close together if initial conditions close.

Chaotic motion: Small changes in initial conditions can lead to large changes in future positions.

Short term predictions are fine, even for chaotic motion. However, long term predictions of chaotic motion is very hard because accuracy needed for initial conditions and sensitivity to small perturbations.

Example: In 1960s a simplified model of weather was shown to be chaotic. This is a property of nonlinear hydrodynamics equations which govern weather forecasting is very hard.

Popular book on Chaos in solar system "Newton's Clock" by E. Peterson, Freeman Co, NY.

Moon rotates stably about its largest moment. Tidal dissipation has locked rotation so that heavier near side always faces Earth.

Amalthea rotates stably about its smallest moment with its long axis pointing to Jupiter.
An irregular moon can start out with an unstable rotation about middle axes. However, expect tidal dissipation to quickly lock spin into a stable rotation about 1st or 3rd axis. This did not happen with Hyperion because Titan keeps kicking it. Hyperion is in a 4:3 resonance with Titan. Titan goes around 4 times for every 3 orbits of Hyperion.

**Euler Angles** \((\alpha, \beta, \gamma)\)

Efficient way of both specifying the orientation of a body w.r.t. some inertial frame and to describe its rotational velocity.

Consider inertial frame fixed w.r.t. Sun.

- \(\hat{e}_z\) is normal to plane of Earth's orbit.
- \(\hat{e}_x\) defines plane of Earth's orbit.
- Let \(\hat{e}_x\) point from Earth to Sun.

North pole of Earth

\(\hat{e}_x\) normal to orbit plane

\(\beta = \theta\)
Earth sits at common origin of $\hat{e}_3$ (inertial) and $\hat{e}_i$ (body fixed of
frame)
$\hat{e}_3$ points to North pole
[ $\hat{e}_i$ and $\hat{e}_3$ are in plane of equator
[ $\hat{e}_i$ is at $\theta$ or latitude, say ]

Let polar angles $\Theta = \beta$ and $\phi = \alpha$

Specify orientation of North pole

$\beta = 23.5^\circ$ Earth's obliquity very important for climate

Need one more angle $\gamma$ to specify orientation of Earth about $\hat{e}_3$ axes,

$\gamma = \omega$ (day $\approx \frac{2\pi}{24}$

Euler angles are defined as a transformation from $\hat{e}_i$ to $\hat{e}_i$.

1. Rotate about $\hat{e}_3$ by $\omega$ to bring $\hat{e}_3$
   into direction of $e_3$ "line of nodes"
   "line of nodes" intersection of orbit plane with plane of equator

2. Rotate about line of nodes by $\beta$
   bringing $\hat{e}_3$ into final orientation

3. Rotate about new $\hat{e}_3$ by $\gamma$.
   This completely determines orientation of body.
\( e_y = e_3 \)
\( e_x = e_1 \sin \beta + e_2 \cos \beta \)
\( e^2_x = e_3 \cos \beta + e_2 \sin \beta \sin \gamma - e_1 \sin \beta \cos \gamma \)
\( \mathbf{w} = \alpha \mathbf{e}_x + \beta \mathbf{e}_y + \gamma \mathbf{e}_z \)

Note that \( \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \) are not orthogonal but still add as vectors.

\( \mathbf{w}_1 = \hat{e}_1 \cdot \mathbf{w} = -\hat{e}_1 \sin \beta \cos \gamma + \hat{e}_2 \sin \beta \sin \gamma \)
\( \mathbf{w}_2 = \hat{e}_2 \cdot \mathbf{w} = \hat{e}_1 \sin \beta \sin \gamma + \hat{e}_2 \cos \gamma \)
\( \mathbf{w}_3 = \hat{e}_3 \cos \beta \mathbf{w} \)

\[ T = \frac{1}{2} \left[ I_1 \mathbf{w}_1^2 + I_2 \mathbf{w}_2^2 + I_3 \mathbf{w}_3^2 \right] \]

General expression is a mess.

Consider symmetric top like Earth with \( I_1 = I_2 \)

\[ T = \frac{1}{2} \left[ I_1 (\omega_1^2 + \omega_2^2) + I_3 \omega_3^2 \right] \]

\[ T = \frac{1}{2} I_1 (\alpha^2 \sin^2 \beta + \beta^2) + \frac{5}{2} I_3 (\alpha \cos \beta + \gamma)^2 \]