10/16/00

Lecture 22 Euler's Equations

Fundamental equation

$$\left(\frac{dL}{dt}\right)_{\text{inertial}} = \Gamma^{(e)}$$

Valid where both L and extenal torque $\Gamma^{(e)}$ calculated about an origin which is

1. fixed in some inertial frame
2. <u>Or</u> located at center of mass of body.

About center of mass L seperates into relative and center of mass components.

$$\left(\frac{dL}{dt}\right)_{\text{inertial}} = \left(\frac{dL}{dt}\right)_{\text{body}} + \omega \wedge L = \Gamma^{e}$$

$$\left(\frac{dL}{dt}\right)_{\text{body}} = -\omega \wedge L + \Gamma^{e}$$

$$L_i = I_i \omega_i \qquad i = 1, 2, 3$$

$$\boxed{\begin{aligned} I_1 \frac{d\omega_1}{dt} &= \omega_2 \omega_3 (I_2 - I_3) + \Gamma_1^{e} \\ I_2 \frac{d\omega_2}{dt} &= \omega_3 \omega_1 (I_3 - I_1) + \Gamma_2^{e} \\ I_3 \frac{d\omega_3}{dt} &= \omega_1 \omega_2 (I_1 - I_2) + \Gamma_1^{e} \end{aligned}}$$

Euler's equations are elegant and compact. The axis 1, 2, 3 tumble with the body. Therefore it can be hard to calculate the $\Gamma_1^{(e)}, \Gamma_2^{(e)}, \Gamma_3^{e}$.

The equations as written are useful in two cases. ① No external torques $\vec{\Gamma}^e = 0$ so all $\Gamma_i^e = 0$ even if you don't know what $\hat{e}_i$ are. ② If axes are fixed

Example Torque free motion

In this case energy

$$E = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

is a constant of motion. Also $\vec{L}$ is conserved but axes $\hat{e}_i(t)$ are time dependent. So instead calculate

$$\vec{L}\cdot\vec{L} = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

which is also a constant of motion

Use E, L^2 to write $\omega_1 = \omega_1[E, L^2, \omega_3]$
$\omega_2 = \omega_2[E, L^2, \omega_3]$

Then

$$\frac{d\omega_3}{dt} = \omega_1\omega_2\frac{I_1 - I_2}{I_3}$$

is an explicit messy single differential equation with an ugly explicit solution in terms of elliptic integrals! This gives $\omega_3(t) \to \omega_1(t),\ \omega_2(t)$

Given these solutions try and determine where $\hat{e}_3(t)$ is pointing. Hard.

Look at a simple case: Symmetric top $I_1 = I_2 \neq I_3$ [Earth to a good approx.]

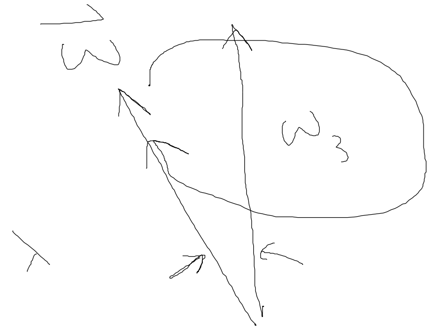
$$\dot{\omega}_3 = 0$$
$$\dot{\omega}_1 = -\Omega\,\omega_2$$
$$\dot{\omega}_2 = \Omega\,\omega_1$$
$$\Omega = \omega_3\,\frac{I_3 - I_1}{I_1}$$

Euler's equations

$\Gamma_i^e = 0$

$I_1 = I_2$

$$\ddot{\omega}_1 = -\Omega^2\,\omega_1$$
$$\omega_3 = \omega\cos\lambda$$
$$\omega_1(t) = \omega\sin\lambda\,\cos\Omega t$$
$$\omega_2(t) = \omega\sin\lambda\,\sin\Omega t$$
$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$$



For Earth $\quad \dfrac{I_3 - I_1}{I_1} \approx \dfrac{1}{305}$

$\Omega \approx \dfrac{\omega}{305}$ period $\sim$ 305 days

Chandler's period

but $\lambda \sim 6\times10^{-7}$ rad

$\vec{L}$ displaced from north pole ~ 4m

Asymmetric top Small amplitude osc.

$I_1 \neq I_2 \neq I_3$

assume motion primarily about third axis

$$\omega_1 \approx \eta_1(t) \ll 1$$
$$\omega_2 \approx \eta_2(t) \ll 1$$
$$\omega_3 \approx \omega_0 + \eta_3(t) \qquad \eta_3(t) \ll 1$$

$$I_1 \dot{\eta}_1 = -(I_2 - I_3)\,\omega_0 \eta_2$$
$$I_2 \dot{\eta}_2 = (I_3 - I_1)\,\omega_0 \eta_1$$
$$I_3 \dot{\eta}_3 = O(\eta_1 \eta_2) \approx 0$$

$$\ddot{\eta}_1 = -\Omega_0^2 \eta_1$$
$$\ddot{\eta}_2 = -\Omega_0^2 \eta_2$$
$$\Omega_0^2 = \omega_0^2 \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}$$

Motion is stable if $\Omega_0^2 > 0$

Thus either $I_3 > I_1$ and $I_3 > I_2$

or

$I_3 < I_1$ and $I_3 < I_2$

Stable about either largest or smallest moment of inertia

If middle moment of inertia $\Rightarrow$ transfer angular momentum to largest I and ... to smallest I. Recitant body and conserve both L and E.

If motion is unstable ω_1, ω_2 grow exp. with time and quickly invalidate our assumption

$$\eta_1/\omega_0 \ll 1 \qquad \eta_2 \ll \omega_0$$

Thus more complex analysis is needed to find full motion. However correct in assumption that ω_1 and ω_2 don't stay small.

Example rotate book about three different axes.