10/11/00

Lecture 20 Rigid Bodies

From chapter 2

$$\left.\frac{d\vec{r}}{dt}\right|_{\text{inertial}} = \left.\frac{d\vec{r}}{dt}\right|_{\text{body}} + \vec{\omega}\wedge\vec{r}$$

For a particle fixed in the body

$$\left.\frac{d\vec{r}}{dt}\right|_{\text{inertial}} = \vec{\omega}\wedge r$$

$$T = \frac{1}{2}\sum_p m_p (\vec{\omega}\wedge\vec{r}_p)\cdot(\vec{\omega}\wedge\vec{r}_p)$$

$$T = \frac{1}{2}\sum_p m_p [\omega^2 r_p^2 - (\omega\cdot r_p)^2]$$

Define moment of inertia tensor

$$I_{ij} = \sum_p m_p (\delta_{ij} r_p^2 - x_{pi} x_{pj})$$

$$\boxed{I_{ij} = \int d^3r\, \rho(r)(\delta_{ij} r^2 - x_i x_j)}$$

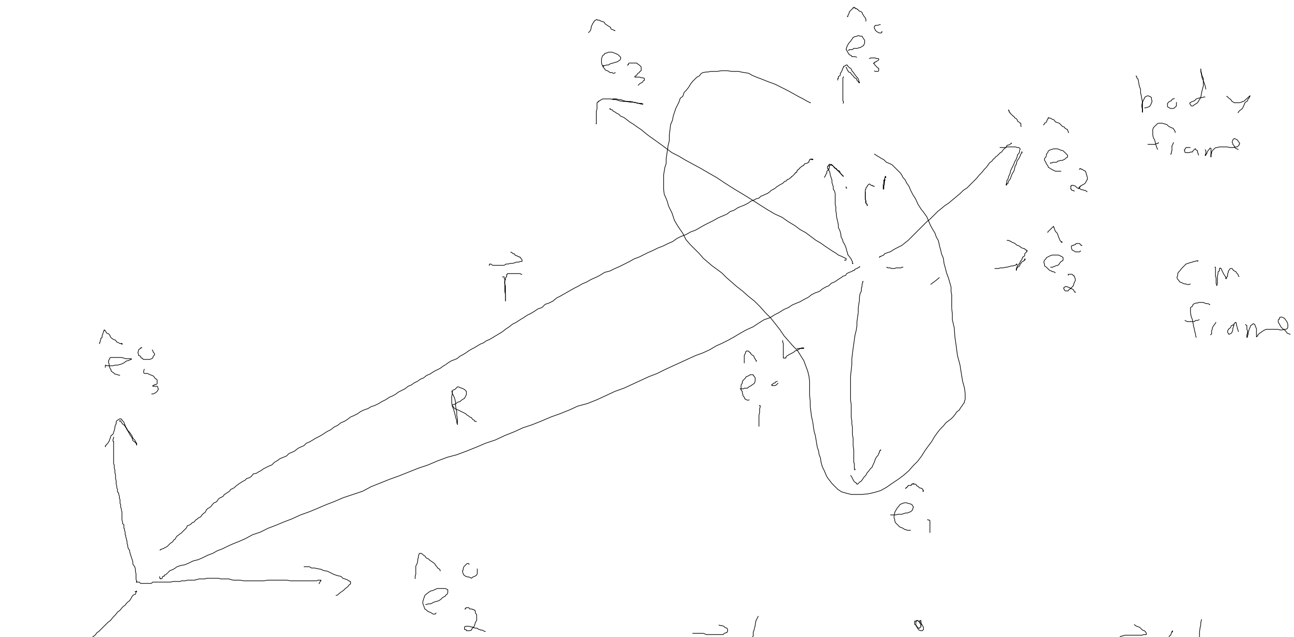
$$T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

$$L = \sum_p m_p \vec{r}_p\wedge\vec{v}_p = \sum_p m_p r_p\wedge(\vec{\omega}\wedge\vec{r}_p)$$

$$L_i = \sum_j I_{ij}\omega_j$$

$$T = \frac{1}{2}\vec{L}\cdot\vec{\omega}$$

General Motion



$$\left.\frac{d\vec{r}}{dt}\right|_{\text{inertial}} = \mathring{R} + \left.\frac{d\vec{r}'}{dt}\right|_{\text{inertial}}$$

Inertial frame

$$\left(\frac{dr'}{dt}\right)_{\text{inertial}} = \left(\frac{dr'}{dt}\right)_{cm} = \sum_{i=1}^{3} \hat{e}_i^c \frac{d}{dt}(\vec{r}' \cdot \hat{e}_i^c)$$

$$\left(\frac{d\vec{r}'}{dt}\right)_{cm} = \left(\frac{d\vec{r}'}{dt}\right)_{body} + \vec{\omega} \wedge \vec{r}'$$

$$\left(\frac{dr}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{r}'}{dt}\right)_{body} + \mathring{R} + \vec{\omega} \wedge \vec{r}'$$

$$T = \tfrac{1}{2} M \mathring{R}^2 + T'$$

$$L = \vec{R} \times (M\mathring{R}) + L'$$

$$L' = \sum_p m_p r'_p \wedge \left(\frac{dr'_p}{dt}\right)_{\text{inertial}} = \sum_p m_p \vec{r}'_p \wedge \left(\frac{d\vec{r}'_p}{dt}\right)_{cm}$$

$$\left(\frac{dL'}{dt}\right)_{\text{inertial}} = \left(\frac{dL'}{dt}\right)_{cm} = \sum_p r_p' \wedge F_p'^{(e)} = \Gamma^{(e)\prime} \quad \text{A}$$

also cm moves

$$M\ddot{R} = \sum_p F_p^{(e)} = F^{(e)} \quad \text{B}$$

General rigid body motion

Torque about cm gives $\frac{dL'}{dt}$

Net external force gives $\ddot{R}$

Moment of inertia tensor

$$\bar{I}_{ij} = \int d^3x \; \rho(x)\, \delta_{ij}(x^2 - x_i x_j) = \sum_p m_p(\delta_{ij}|x_p|^2 - x_{pi}x_{pj})$$

$\bar{I}_{ij} = I_{ij}$ in body fixed frame with origin at center of mass

Parallel axis theorem

$$\vec{y} = \vec{x} - \vec{a}$$

$$I_{ij} = \bar{I}_{ij} + M(a^2\delta_{ij} - a_i a_j)$$

Just substitute $\vec{x} = \vec{y} + \vec{a}$ in integral for $\bar{I}_{ij}$

Note $\int d^3x \; \rho(x)\, \vec{x} = 0$ in cm frame

<u>Principal Axes</u> Chose body fixed frame so that I_{ij} is diagonal

$$I_{ij} = I_i \delta_{ij}$$

Chose directions $\hat{e}_j$

$$\sum_j I_{ij}\, \hat{e}_j = \lambda\, e_i$$

or
$$\det |I_{ij} - \lambda \delta_{ij}| = 0$$
gives three eigenvalues $\lambda_i = I_i$

$e_s^{(i)}$ corresponding eigen vector

Modal matrix

$$\underline{A} = \begin{bmatrix} e^{(1)} & e^{(2)} & e^{(3)} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$A^T A = A^{-1} A = 1$$

For $\underline{m} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ unit matrix instead of mass matrix

$$\underline{e}^{(s)T} \underline{e}^{(m)} = \delta_{sm}$$

$$A^T I A = \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix}$$

The $e^{(s)}$ are principal axes

$\underline{\xi}$ new vector $\underline{\omega} = \underline{A}\, \underline{\xi}$

or $\xi = A^{-1} \omega = A^T \omega$

$$\underline{\xi} = \begin{bmatrix} e^{(1)} \cdot \omega \\ e^{(2)} \cdot \omega \\ e^{(3)} \cdot \omega \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

Components of ω along principal axes

$$T = \tfrac{1}{2} \underline{\omega}^T \underline{I}\, \underline{\omega} = \tfrac{1}{2} \underline{\xi}^T A^T I A \xi$$
$$= \tfrac{1}{2} \sum \xi_s^2 I_s$$

$$L_{new} = A^T L = A^T I \omega = A^T I A \xi =$$

$$= \sum_s I_s \xi_s$$

$$I_s = (A^T I A)_{ss} = \int d^3r \, \rho(r) \left[r^2 - (\vec{r} \cdot \hat{e}^{(s)})^2 \right]$$

Example (5.7) Gravitational Pot. of Earth

$$\Phi(\vec{r}) = -G \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

Multipole expansion

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta_{rr'})$$

also can use spherical addition theorem

$$P_l(\cos\theta_{rr'}) = \left(\frac{4\pi}{2l+1}\right) \sum_{m=-l}^{l} Y^*_{lm}(\theta, \phi) \, Y_{lm}(\theta', \phi')$$

$$\Phi(\vec{r}) = \Phi_0 + \Phi_1 + \Phi_2 + \ldots$$

$$\Phi_0 = -\frac{GM}{r}$$

$\Phi_1 = 0$ if origin chosen at cm of Earth

$$\Phi_2(\vec{r}) = -\frac{G}{r^3} \left(\frac{4\pi}{5}\right) \sum_m Y^*_{2m}(\theta, \phi) \, Q_{2m}$$

$$Q_{2m} \equiv \int d^3r' \, r'^2 \rho(\vec{r}') \, Y_{2m}(\theta', \phi')$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} P_2(\cos\theta) \qquad P_2(x) = \frac{3x^2 - 1}{2}$$

If mass distribution of Earth is independent of ϕ. [Chose z axis to correspond to rotation axis.]

$$Y_{2m} \sim e^{im\phi}$$

$$\int_0^{2\pi} d\phi \; e^{im\phi} = 0 \qquad m \neq 0$$

Therefore $\quad Q_{2m} = 0 \qquad m \neq 0$

$$\Phi_2(\vec{r}) = -\frac{G}{r^3} P_2(\cos\theta) \; q_2$$

$$q_2 \equiv \int d^3r' \, \rho(r') \, r'^2 \, P_2(\cos\theta')$$

$$q_2 = \int d^3r' \, \rho(r') \left[\frac{3}{2} z'^2 - \frac{r'^2}{2}\right]$$

Compare with $\quad I_3 = \int d^3r' \, \rho(r') \, (r'^2 - z'^2)$

$$I_1 = I_2 = \int d^3r' \, \rho(r') \left(r'^2 - \frac{x'^2+y'^2}{2}\right)$$

$$I_1 = \int d^3r' \, \rho(r') \left(\frac{r'^2}{2} + \frac{z'^2}{2}\right)$$

Compare $\quad \boxed{q_2 = I_1 - I_3}$

$$\Phi_2 = -\frac{G}{r^3} (I_1 - I_3) \, P_2(\cos\theta)$$

or

$$\Phi_2 = -\left(\frac{GM_E}{r}\right) \left(\frac{I_1 - I_3}{M_E R^2}\right) \left(\frac{R_E}{r}\right)^2 P_2(\cos\theta)$$

So $\quad \Phi \simeq -\frac{GM_E}{r}\left[1 - J_2 \left(\frac{R_e}{r}\right)^2 P_2(\cos\theta)\right]$

$$J_2 = (I_3 - I_1) / M_E R^2$$