10/9/00

Lecture 19 Continuous String

Lagrangian for transverse osc. of springs

$$L = \sum_{i=1}^{N}\left[\frac{m}{2}\dot{\mu}_i^2 - \frac{\tau}{2a}(\mu_{i+1}-\mu_i)^2\right]$$

Now take continuum limit

$$a \to 0 \qquad \mu_i(t) \to U(x,t)$$

$\frac{m}{a} = \sigma$ mass density per unit length

$$\sum_i a = \sum_i \Delta x \longrightarrow \int_0^{\ell} dx$$

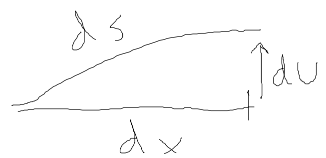
$$\frac{\mu_{i+1}-\mu_i}{a} = \frac{\partial}{\partial x}U(x,t)$$

$$L = \frac{m}{2a}\sum_i a\,\dot{\mu}_i^2 - \frac{\tau}{2}\sum_i a\left(\frac{\mu_{i+1}-\mu_i}{a}\right)^2$$

$$\longrightarrow \frac{\sigma}{2}\int_0^{\ell} dx\left[\frac{\partial U}{\partial t}\right]^2 - \frac{\tau}{2}\int dx\left(\frac{\partial U}{\partial x}\right)^2$$

Kinetic energy is just $\int dx\, \frac{\sigma}{2} v^2$

Pot. energy involves work to strech string



$$dW = \tau(ds - dx) = \tau\, dx\left[\left\{1 + \left(\frac{\partial U}{\partial x}\right)^2\right\}^{1/2} - 1\right]$$

$$\simeq \frac{\tau}{2}\left[\frac{\partial U}{\partial x}\right]^2 dx$$

$$V = \int dW = \int_0^l \frac{\tau}{2}\left[\frac{\partial u}{\partial x}\right]^2 dx$$

$$\boxed{L = \int_0^l dx \left[\frac{\sigma}{2}\left[\frac{\partial u}{\partial t}\right]^2 - \frac{\tau}{2}\left[\frac{\partial u}{\partial x}\right]^2\right]}$$

Hamilton's Principle for Continuous System

$$\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} dt \int_0^l dx\, \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}; x, t\right) = 0$$

$\mathcal{L}$ = Lagrangian density

$$= \frac{\sigma}{2}\left[\frac{\partial u}{\partial t}\right]^2 - \frac{\tau}{2}\left[\frac{\partial u}{\partial x}\right]^2 \qquad \text{for string}$$

String with fixed end points

$$\delta u(x=0, t) = \delta u(x=l, t) = 0 \qquad \text{for all } t$$

also

$$\delta u(x, t_1) = \delta u(x, t_2) = 0 \qquad \text{for all } x$$

$$\delta \int L\, dt = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial u}\delta u + \frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)}\delta \frac{\partial u}{\partial x} + \frac{\partial \mathcal{L}}{\partial \frac{\partial u}{\partial t}}\delta \frac{\partial u}{\partial t}\right] = 0$$

Integrate by parts and use b. conditions to get rid of surface terms.

$$\delta \int L\, dt = \int dt \int dx \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial u}{\partial t}}\right]\delta u = 0$$

Should be true for general δu

Lagranges Eqs for cont. system

$$\boxed{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial u}{\partial t}} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial u}{\partial x}}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0}$$

Example uniform string

$$\mathcal{L} = \frac{\sigma}{2}\left[\frac{\partial u}{\partial t}\right]^2 - \frac{\tau}{2}\left[\frac{\partial u}{\partial x}\right]^2$$

$$\frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\sigma \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \tau/\sigma$$

Speed of wave $c = \sqrt{\tau/\sigma}$

1-dim wave eq.

Note discreate equations of motion

$$m\ddot{\mu}_i + \frac{2\tau}{a}\mu_i - \frac{\tau}{a}(\mu_{i+1} + \mu_{i-1}) = 0$$

$$i = 1, \dots, N$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\mu_{i+1} + \mu_{i-1} - 2\mu_i}{a^2}$$

$$m\frac{\partial^2 u}{\partial t^2} - \tau a \frac{\partial^2 u}{\partial x^2} \approx 0$$

$$\frac{m}{\tau a}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \checkmark \qquad \frac{m}{a} = \sigma \qquad \frac{\sigma}{\tau} = \frac{1}{c^2}$$

General Solution to wave Equation

Normal modes

$$u(x,t) = C\,\rho(x)\cos(\omega t+\phi)$$

as before $\rho_i \longrightarrow \rho(x)$

$$\frac{d^2\rho}{dx^2} + k^2\rho = 0 \qquad k \equiv \frac{\omega}{c}$$

General solution $\rho = \sin kx$ or $\cos kx$

B.C. $\rho(0)=0 \longrightarrow \sin kx$

$$\rho(x) = \left(\frac{2}{l\sigma}\right)^{1/2}\sin kx$$

Normalization

$$\rho^{(n)T}\,\underline{m}\,\rho^{(n)} = 1$$

$$\Rightarrow \int_0^l dx\,\rho^2(x)\,\sigma = 1$$

2nd b.c. $\rho(l)=0$

$$k = k_n = \frac{n\pi}{l}$$

$$\rho_n(x) = \left(\frac{2}{l\sigma}\right)^{1/2}\sin\left(\frac{n\pi x}{l}\right)$$

$$\omega_n = c k_n = \left(\frac{\tau}{\sigma}\right)^{1/2}\frac{n\pi}{l}$$

Note index n is discrete. Normal mode frequency ω_n are discrete even for a continuous string. Only difference is sum over normal modes goes to ∞. Before $n \leq N$ for discrete spring system

Linear superposition of normal modes provides general solution

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n(x)\, C_n \cos(\omega_n t + \phi_n)$$

$$\zeta_n(t) = C_n \cos(\omega_n t + \phi_n)$$

$$u(x,t) = \sum_n \rho_n(x)\, \zeta_n(t)$$

L involves $V = -\frac{\tau}{2} \int_0^{\ell} u(x,t) \frac{\partial^2 u}{\partial x^2}\, dx$

$$\frac{\partial^2 \rho_n}{\partial x^2} = -k_n^2 \rho_n \qquad k_n = \frac{\omega_n}{c} = \frac{n\pi}{\ell}$$

Use $\int_0^{\ell} dx\, \rho_n(x)\, \sigma\, \rho_m(x) = \delta_{nm}$

$$L = \frac{1}{2} \sum_{n=1}^{\infty} \left[\dot{\zeta}_n^2 - \omega_n^2 \zeta_n^2 \right]$$

Looks like discrete system except sum goes to ∞