Lecture 19 Continuous String

Lagrangian for transverse osc. of springs

\[ L = \sum_{i=1}^{n} \left[ \frac{m_i}{2} u_i^2 - \frac{c}{2a} (M_{i+1} - M_i)^2 \right] \]

Now to continuous limit

\[ a \to 0 \quad M_i(t) \to u(x,t) \]

\[ \frac{m}{a} = \sigma \quad \text{mass density per unit length} \]

\[ S \sum_{i=1}^{l} a \to \int_0^l dx \]

\[ \frac{M_{i+1} - M_i}{a} = \frac{\partial u}{\partial x} \]

\[ L = \frac{m}{2a} \sum_{i=1}^{l} a \frac{du^2}{dt} - \frac{c}{2} \sum_{i=1}^{l} a \left( \frac{M_{i+1} - M_i}{a} \right)^2 \]

\[ \Rightarrow \frac{\sigma}{2} \left[ \frac{d}{dx} \left( \frac{\partial u}{\partial t} \right) \right]^2 - \frac{c}{2} \int_0^l dx \left( \frac{\partial u}{\partial x} \right)^2 \]

Kinetic energy is just \( Sdx \frac{\sigma}{2} V^2 \)

Pot. energy involves work to stretch string

\[ dW = \frac{c}{2} (dx - ds) = \frac{\sigma}{2} dx \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \right] \]

\[ = \frac{\sigma}{2} \left[ \frac{\partial u}{\partial x} \right]^2 dx \]
\[ V = \int_{-l}^{l} - \frac{1}{2} \left[ \frac{\partial u}{\partial x} \right]^2 \, dx \]

\[ L = \int_{-l}^{l} \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} \right) \right]^2 \, dx \]

Hamilton's Principle for Continuous System

\[ \delta \int_{t_1}^{t_2} L \, dt = 0 \]

\[ L = \text{Lagrangian density} \]

\[ = \frac{\alpha}{2} \left[ \frac{\partial u}{\partial x} \right]^2 - \frac{\kappa}{2} \left[ \frac{\partial u}{\partial t} \right]^2 \]

for string

String with fixed end points

\[ \delta u(x=0, t) = \delta u(x=l, t) = 0 \]

for all \( t \)

also

\[ \delta u(x, t_1) = \delta u(x, t_2) = 0 \]

Integrate by parts and use boundary conditions to get rid of surface terms.

\[ \int_{t_1}^{t_2} \delta L \, dt = \int_{-l}^{l} \left[ \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} \right) \right]^2 \, dx \]

Should be true for general \( u \).
Languages

Eqs... for cont. system

\[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \left( \frac{\partial f}{\partial x} \right) = 0 \]

Example: Uniform string

\[ T = \frac{E}{2} \left[ \frac{\partial u}{\partial t} \right]^2 - \frac{\rho}{2} \left[ \frac{\partial u}{\partial x} \right]^2 \]

\[ \frac{\partial f}{\partial u} = 0 \]

\[ 6 \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x^2} = 0 \]

\[ \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{1}{6} \]

Speed of wave \( c = \sqrt{\frac{1}{6}} \)

1-dim wave eq.

Note: Discrete equations of motion

\[ m \frac{\ddot{u}_i}{a} + 2 \frac{\ddot{u}_i}{a} M_i - \frac{c}{a} (M_{i+1} + M_{i-1}) = 0 \]

\[ i = 1, \ldots, N \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{M_{i+1} + M_{i-1} - 2M_i}{a^2} \]

\[ m \frac{\ddot{u}_i}{a} - \frac{c}{a} \frac{\partial^2 u}{\partial x^2} \]

\[ \frac{m}{c^2} \frac{\ddot{u}_i}{a} = \frac{\partial^2 u}{\partial x^2} \quad \text{sqrt} \quad \frac{m}{a} = c \quad \frac{c}{c^2} = \frac{1}{c^2} \]
General Solution to Wave Equation

Normal modes

\[ u(x,t) = C \ p(x) \ \cos(\omega t + \phi) \]

As before \( p_i \rightarrow p(x) \)

\[ \frac{d^2 p}{dx^2} + k^2 p = 0 \quad k = \frac{\omega}{c} \]

General solution \( p = \sin kx \) or \( \cos kx \)

B.C. \( p(0) = 0 \rightarrow \sin kx \)

\[ p(x) = \left( \frac{2}{l^6} \right)^{\frac{1}{2}} \ \sin kx \]

Normalization

\[ \int_0^l p^{(m)} T \ dx \ p^{(m)} = 1 \]

\[ \Rightarrow \int_0^l p^2(x) \ dx = 1 \]

2nd b.c. \( p(l) = 0 \)

\[ k = k_n = \frac{n\pi}{l} \]

\[ p_n(x) = \left( \frac{2}{l^6} \right)^{\frac{1}{2}} \ \sin \left( \frac{n\pi x}{l} \right) \]

\[ \omega_n = c k_n = \left( \frac{2}{l^6} \right)^{\frac{1}{2}} \ \frac{n\pi}{l} \]
Note: index $n$ is discrete. Normal mode frequencies $\omega_n$ are discrete even for a continuous string. Only difference is that the sum over normal modes goes to $\infty$.

Linear superposition of normal modes provides the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n \phi_n \cos(\omega_n t + \varphi_n)$$

$$\tau_n(t) = C_n \cos(\omega_n t + \varphi_n)$$

$$v(x,t) = \sum_{n=1}^{\infty} \rho_n \phi_n \tau_n(t)$$

$L$ involves:

$$V = -\frac{1}{2} \int_0^L \frac{\partial^2 u}{\partial x^2} \, dx$$

$$\frac{\partial^2 \rho_n}{\partial x^2} = -k_n \rho_n \quad k_n = \frac{\omega_n^2}{L}$$

Use:

$$\int_0^L \rho_n(x) \, dx \phi_m(x) = \delta_{nm}$$

$$L = \frac{1}{2} \sum_{n=1}^{\infty} \left( \delta_n \omega^2_n - \omega_n \gamma_n^2 \right)$$

Looks like discrete system except the sum goes to $\infty$. 