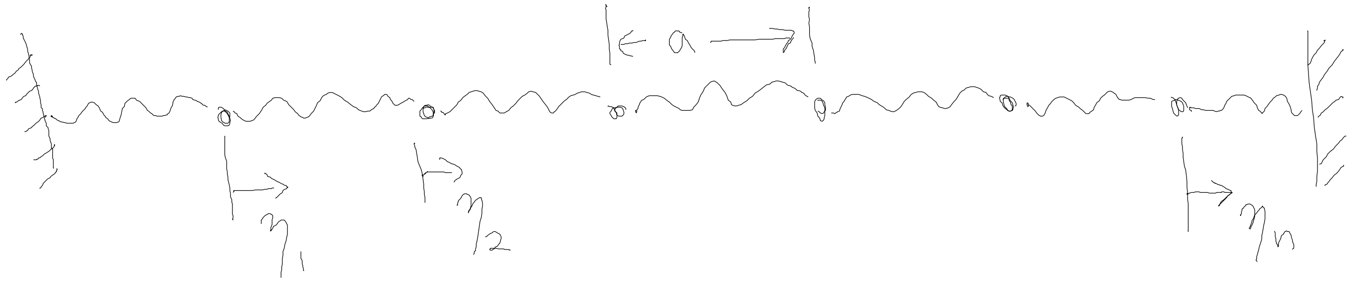
10/6/00

# Lecture 18 Springs → Strings

Example of N body problem




Consider N masses (all m) connected by N+1 springs.

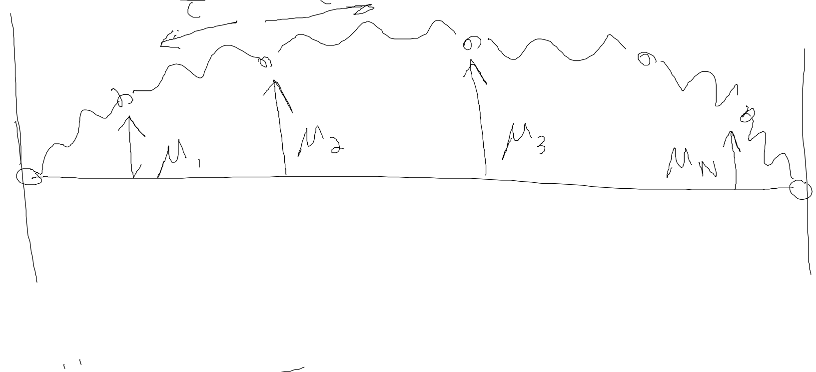
Define $\eta_0 = \eta_{N+1} \equiv 0$ (fixed ends)

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i^2 - \frac{k}{2} \sum_{i=0}^{N} (\eta_{i+1} - \eta_i)^2$$

Lagrange's equations $0 = m\ddot{\eta}_i - k(\eta_{i+1} - \eta_i) + k(\eta_i - \eta_{i-1})$

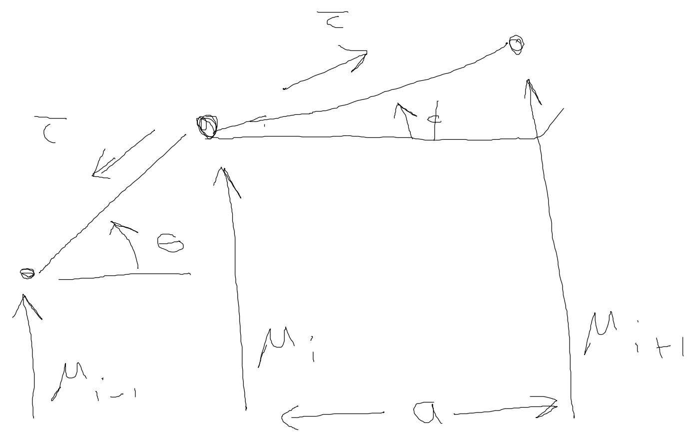
$$m\ddot{\eta}_i + 2k\eta_i - k(\eta_{i+1} + \eta_{i-1}) = 0 \qquad i = 1, \dots, N$$

Another version of same problem. Transverse osc. of stretched springs




$$m\ddot{\mu}_i = F_i = \tau \sin\phi - \sin\theta \simeq \tau \phi - \theta$$

two springs pull in slightly different directions




$$m\ddot{\mu}_i = \frac{2\tau}{a}\mu_i - \frac{\tau}{a}(\mu_{i+1} + \mu_{i-1})$$

Fixed ends $\quad \mu_0 = \mu_{N+1} \equiv 0$

$$L = T - V = \frac{1}{2} m \sum_i \dot{\mu}_i^2 - \frac{\tau}{2a}\sum_i (\mu_{i+1} - \mu_i)^2$$

$$\eta_i \to \mu_i \quad \text{and} \quad k \to \frac{\tau}{a}$$

Lagrangian has same form. Note in 3 dim have Longitudinal modes and then two uncoupled sets of transverse modes for the two transverse directions

Normal mode solutions

$$\mu_i(t) = C\rho_i \cos(\omega t + \phi) \qquad i = 1, \ldots, N$$

(A) $$\left[\frac{2\tau}{a} - m\omega^2\right]\rho_i - \frac{\tau}{a}(\rho_{i+1} + \rho_{i-1}) = 0$$

let $\lambda = 2 - \frac{m\omega^2 a}{\tau}$

$$\lambda\rho_i - (\rho_{i+1} + \rho_{i-1}) = 0$$

$$\lambda\rho_1 - \rho_2 = 0 \qquad\qquad \rho_0 = 0$$

$$-\rho_1 + \lambda\rho_2 - \rho_3 = 0$$

$$-\rho_2 + \lambda \rho_3 - \rho_4 = 0$$

$$\cdots \qquad -\rho_{N-1} + \lambda \rho_N = 0 \qquad \rho_{N+1} = 0$$

Normal mode frequencies from

$$D_N = \det \begin{vmatrix} \lambda & -1 & & & & 0 \\ -1 & \lambda & -1 & & & \\ & -1 & \lambda & -1 & & \\ & & & \ddots & & \\ & & & -1 & \lambda & -1 \\ 0 & & & & -1 & \lambda \end{vmatrix}$$

Tridiagonal matrix

See text to evaluate $D_N$

Another way is to look for plane waves

$$\mu_j = \mu(x_j, t) = A e^{i(k x_j - \omega t)}$$

Put into eq. (A)

$x_j = j a$ position in lattice

$$-m\omega^2 + \frac{2\tau}{a} - \frac{\tau}{a}\left(e^{ika} + e^{-ika}\right) = 0$$

Solve for $\omega^2$

$$\boxed{\omega^2 = \frac{2\tau}{ma}(1 - \cos ka) = \frac{4\tau}{ma}\sin^2\frac{ka}{2}}$$

This is called a dispersion relation $\omega(k)$

The normal mode frequencies follow from the allowed values of $k$.

Expect $N$ normal modes $\longrightarrow$ $N$ allowed $k$ values.

Allowed $k$ determined by boundary conditions

Fixed ends
$$\mu(x_0) = \mu(0) = 0$$
$$\mu((N+1)a) = 0$$

Consider adding $e^{ikx}$ and $\pm e^{-ikx}$ chose $-$ to satisfy $\mu(0) = 0$

$$\mu(x_j, t) = A\left[e^{ikx_j} - e^{-ikx_j}\right]e^{-i\omega t}$$

Note $\omega$ only a function of $k^2$ not $\omega(k) = \omega(-k)$

By construction $\mu(0) = 0$

$$\mu((N+1)a, t) = 0 = \left(e^{ik(N+1)a} - e^{-ik(N+1)a}\right)$$

$$\Rightarrow \sin k(N+1)a = 0$$

$$\Rightarrow k = \frac{n\pi}{a(N+1)} = k_n$$

$$n = 1, 2, , \ldots N$$

note $n > N$ is not needed it gives nothing new

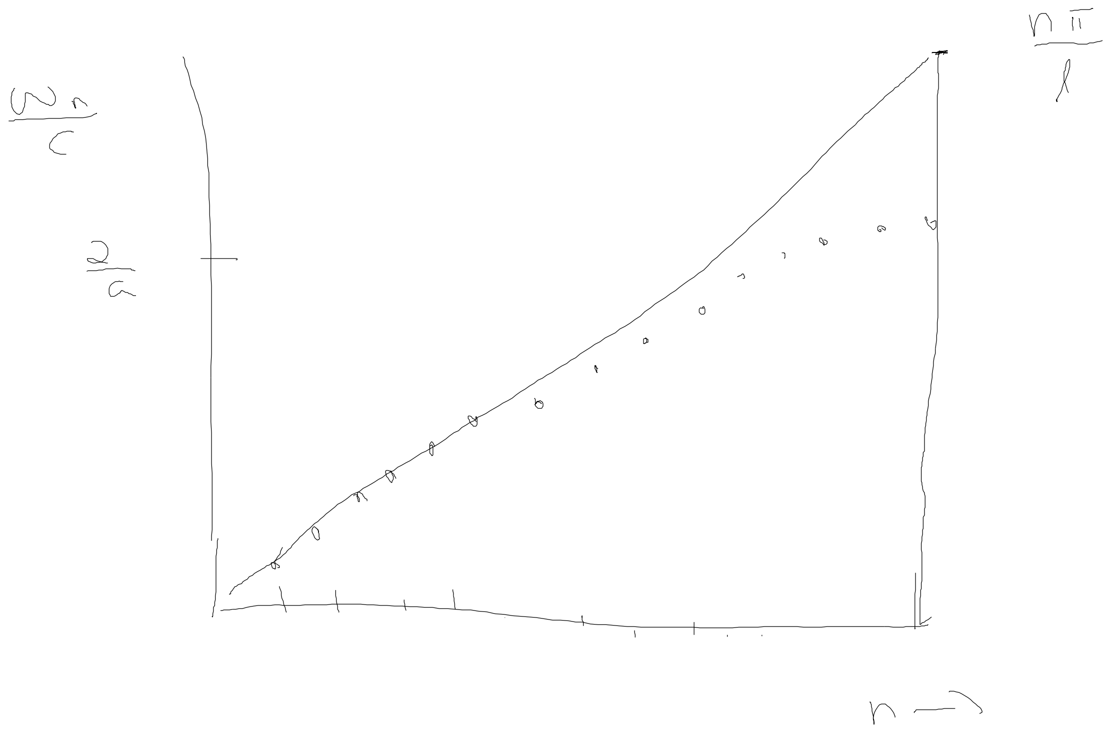
$$\mu(x_j t) = 2iA_n \sin\left(\frac{n\pi x_j}{a N+1}\right) e^{-i\omega_n t}$$

$$\mu_j = 2A_n \sin\left[\frac{n\pi x_j}{a(N+1)}\right] \sin \omega_n t$$

$$\frac{\omega_n}{c} = \frac{2}{a} \sin \frac{n\pi a}{2l}$$

$$c = \left(\frac{T}{m/a}\right)^{1/2} \qquad \text{wave velocity}$$

$$l = (N+1)\,a$$




Wavelength of osc. $k = 2\pi/\lambda$

$$\frac{2\pi}{\lambda}(N+1)\,a = n\pi$$

$$\lambda = \frac{2l}{n} \qquad n = 1, \ldots, N$$

Modes with $n > N$ have $\lambda < 2a$