10/4/00

Lecture 17 Normal Modes Cont.

Modal Matrix

$$\underline{A} \equiv \begin{bmatrix} \rho^{(1)} & \rho^{(2)} & \cdots & \rho^{(n)} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$A_{\lambda\sigma} \equiv \rho_\lambda^{(\sigma)}$$

$$\left[\underline{A}^T \underline{m} \, \underline{A}\right]_{\mu\nu} = \sum_{\lambda\sigma} (A^T)_{\mu\lambda} m_{\lambda\sigma} A_{\sigma\nu}$$

$$= \sum_{\lambda\rho} \rho_\lambda^{(\mu)} m_{\lambda\sigma} \rho_\sigma^{(\nu)} = \delta_{\mu\nu}$$

From orthogonality of $\rho^{(\mu)}, \rho^{(\nu)}$

$$\boxed{\underline{A}^T \underline{m} \, \underline{A} = \underline{1}}$$

$$(\underline{A}^T \underline{v} \, \underline{A})_{\mu\nu} = \sum_\lambda \rho_\lambda^{(\mu)} \sum_\sigma v_{\lambda\sigma} \rho_\sigma^{(\nu)} = \rho_\lambda^{(\mu)} \sum_\sigma \omega_\nu^2 m_{\lambda\sigma} \rho_\sigma^{(\nu)}$$

Remember $(\underline{v} - \omega_s^2 \underline{m}) \rho^{(s)} = 0$

and $\det |\underline{v} - \omega^2 \underline{m}) = 0$

$$(\underline{A}^T \underline{v} \, \underline{A}) = \omega_\nu^2 \delta_{\mu\nu}$$

$$\underline{\omega}_D^2 = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \ddots \\ & & & \omega_n^2 \end{bmatrix} \qquad \underline{A}^T \underline{v} \, \underline{A} = \underline{\omega}_D^2$$

$\underline{A}$ diagonalizes both $\underline{V}$ and $\underline{m}$ at same time

Normal Coordinates

Define a new set of coordinates

$$\eta(t) \equiv \underline{A}\, \underline{\xi}(t)$$

Invert by multiplying by $\underline{A}^T \underline{m}$ from left

$$\underline{A}^T \underline{m}\, \eta(t) = \underline{A}^T \underline{m}\, \underline{A}\, \xi(t) = \xi(t)$$

This defines a new set of coordinates in wich the Lagrangian is diagonal!

$$L = \frac{1}{2} \dot{\eta}^T \underline{m}\, \dot{\eta} - \frac{1}{2} \eta^T \underline{V}\, \eta \qquad \text{original } L$$

$$L = \frac{1}{2} \sum_{\lambda\sigma} \left(m_{\lambda\sigma} \dot{\eta}_\lambda \dot{\eta}_\sigma - V_{\lambda\sigma} \eta_\lambda \eta_\sigma \right)$$

$$\dot{\eta}^T = \left[\underline{A}\, \dot{\underline{\xi}} \right]^T$$

but $\underline{A}$ is a constant matrix.

$$= \dot{\xi}^T A^T \qquad \left[\underline{A}\underline{B}\right]^T = \underline{B}^T \underline{A}^T$$

$$L = \frac{1}{2} \dot{\xi}^T A^T m A \dot{\xi} - \frac{1}{2} \xi^T A^T V A \xi$$

$$= \frac{1}{2} \dot{\xi}^T \dot{\xi} - \frac{1}{2} \xi^T \omega_D^2 \xi$$

$$L = \frac{1}{2} \sum_\sigma \left(\dot{\xi}_\sigma^2 - \omega_\sigma^2 \xi_\sigma^2 \right)$$

Reduced problem to uncoupled simple harmonic motion.

Note procedure applies to any system no matter how coupled and complicated provided amplitude of motion is small

Lagrange's equations

$$\ddot{\xi}_\sigma = -\omega_\sigma^2 \xi_\sigma$$

$$\xi_\sigma = C^{(\sigma)} \cos(\omega_\sigma t + \phi_\sigma) \qquad \text{general solution}$$

$$\eta_\lambda(t) = \sum_{s=1}^{n} \rho_\lambda^{(s)} \cos(\omega_s t + \phi_s)$$

From $\quad \underline{\eta} = \underline{A}\, \underline{\xi}$

Each normal mode undergoes simple harmonic motion at frequency ω_σ

Remember Amplitudes $C^{(\sigma)}$ and phases ϕ_σ determined by initial conditions

$$\eta_\lambda(0) = \sum_s \rho_\lambda^{(s)} \cos\phi_s$$

$$\dot{\eta}_\lambda(0) = -\sum_s \omega_s \rho_\lambda^{(s)} \sin\phi_s$$

$$\underline{\rho}^{(t)T}\, \underline{m}\, \underline{\eta}(0) = \sum_s C^{(s)} \rho^{(t)T} m\, \rho^{(s)} \cos\phi_s = C^{(t)} \cos\phi_t$$

$$\underline{\rho}^{(t)T}\, \underline{m}\, \dot{\underline{\eta}}(0) = -C^{(t)} \omega_t \sin\phi_t$$

Remember general procedure

$x_i(t)$ = original Cartesian coordinates

$q_\lambda(t)$ = generalized coordinates

$$q_\lambda(t) = q_\lambda^0 + \eta_\lambda(t)$$

At q_λ^0 all generalized forces $-\left.\dfrac{\partial V}{\partial q_\sigma}\right|_{q^0} = 0$ vanish

Then look for small amplitude motion about q^0_σ
Assume $\eta_\lambda(t)$ are small and expand Lagrangian to 2^{nd} order in $\eta_\lambda, \dot{\eta}_\lambda$

$$L = \frac{1}{2} \sum_{\sigma\lambda} m_{\sigma\lambda} \dot{\eta}_\sigma \dot{\eta}_\lambda - \frac{1}{2} \sum_{\sigma\lambda} V_{\sigma\lambda} \eta_\sigma \eta_\lambda$$

$$m_{\sigma\lambda} = \sum_i m_i \frac{\partial x}{\partial q_\sigma} \frac{\partial x}{\partial q_\lambda}\Bigg|_{q^0} \qquad V_{\sigma\lambda} = \frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}\Bigg|_{q^0}$$

Then go to normal mode coordinates

$$\eta(t) = \underline{A}\ \underline{\zeta}(t)$$

or

$$\zeta(t) = \underline{A}^T\ \underline{m}\ \eta(t)$$

In the new coordinates motion is just a collection of n uncoupled harmonic osc.

Can have arbitrary amplitude in each osc. Chose amplitudes to satisfy initial conditions

$$C^{(\sigma)} \cos\phi_\sigma = \zeta_\sigma(0)$$

$$\dot{\zeta}_\sigma(0) = -\omega_\sigma C^{(\sigma)} \sin\phi_\sigma$$