11/29/00

# Elliptical Drum  Lecture 39

Last time

$$R = a\left[1 + \sum_{p=2}^{\infty} (\epsilon_p \cos p\phi + \bar{\epsilon}_p \sin p\phi)\right]$$

① No change in frequency to 1st order in $\epsilon_p, \bar{\epsilon}_p$.

② $$\rho(r,\phi) = J_0(kr) + \sum_m (A_m \cos m\phi + B_m \sin m\phi) J_m(kr)$$

This satisfies

$$(\nabla^2 + k^2)\rho = 0$$

and

$$\rho(r = R(\phi), \phi) = 0$$

$\Rightarrow$

$$A_m = \epsilon_m ka\, J_1(ka) / J_m(ka)$$
$$B_m = \bar{\epsilon}_m ka\, J_1(ka) / J_m(ka)$$

$$J_0(ka) = 0 \quad \Rightarrow \quad k = \alpha_{0n}/a$$
$$\omega_{0n} = \alpha_{0n} c/a$$

Lowest mode $\quad \dfrac{\omega_{01}}{c} = \dfrac{2.4048}{a} = \dfrac{2.4048\,\pi^{1/2}}{A^{1/2}} = \boxed{\dfrac{4.2624}{A^{1/2}}}$

$A = \pi a^2$ area of membrane

Square membrane $\quad \dfrac{\omega_{11}}{c} = \dfrac{(2\pi^2)^{1/2}}{A^{1/2}} = \dfrac{4.4429}{A^{1/2}}$

Square drum is only 4% higher in frequency than circular drum.

Now prove circular drum has lowest freq. (for given area) of any shaped drum.

$2^{nd}$ order calculation of $\omega_{on}$ see Problem 8.12 of F+W

Expand $J_0(kR)$ to $2^{nd}$ order

$$J_0(kR) \simeq J_0(ka) + k(R-a)J_0'(ka) + \frac{[k(R-a)]^2}{2} J_0''(ka)$$

$$J_0' = -J_1$$

$$J_0'' = \frac{1}{ka}J_1 - J_0$$

Only need to expand $J_m(kR)$ to first order in $R-a$ because $A_m, B_m$ are all ready $1^{st}$ order.

$$J_m(kR) \simeq J_m(ka) + k(R-a)J_m'(ka)$$

$$\rho(R,\phi) = 0 = J_0(ka) - ka\,J_1(ka)\sum_p(\epsilon_p c_p + \bar{\epsilon}_p s_p)$$
$$+ \frac{(ka)^2}{2}\left[\sum_p(\epsilon_p c_p + \bar{\epsilon}_p s_p)\right]^2 J_0''(ka)$$
$$+ \sum_m J_m(ka)\left[\epsilon_m c_m \frac{J_1}{J_m} ka + \bar{\epsilon}_m s_m \frac{J_1}{J_m} ka\right]$$
$$+ \sum_m J_m'(ka)\, ka \sum_p(\epsilon_p c_p + \bar{\epsilon}_p s_p)\, \underbrace{ka\frac{J_1(ka)}{J_m(ka)}\left[\epsilon_m c_m + \bar{\epsilon}_m s_m\right]}_{A_m c_m + B_m s_m}$$

Here $c_p = \cos p\phi$, $s_p = \sin p\phi$

To solve for $k$ project out $m=0$ term in F.S. in $\phi$.

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \; \rho(R,\phi) = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \; c_p c_q = \frac{1}{2}\delta_{pq} = \frac{1}{2\pi}\int d\phi \; s_p s_q$$

$$\int d\phi \; c_p s_q = 0$$

$$J_0(ka) + \frac{(ka)^2}{2} J_0''(ka) \; \frac{1}{2} \sum_p \left(\epsilon_p^2 + \bar{\epsilon}_p^2\right)$$

$$+ (ka)^2 \sum_m \frac{J_1(ka)\, J_m'(ka)}{J_m(ka)} \; \frac{1}{2}\left(\epsilon_m^2 + \bar{\epsilon}_m^2\right) = 0$$

$$J_0(ka) \simeq J_0(k_0 a) + (k-k_0)a \, J_0'(k_0 a)$$

$$= J_0(k_0 a) - (k-k_0) a \, J_1(k_0 a)$$

Here $k_0 a = \alpha_{0n}$ is lowest order result. and $ka$ is 2nd order.

Note $(k-k_0)a$ is order $\epsilon_p^2$

In all the other terms can just replace $ka$ by $k_0 a$ because they are all ready order $\epsilon_p^2$.

$$J_0(k_0 a) = 0$$

$$J_0''(k_0 a) \simeq J_0''(k_0 a) = \frac{1}{k_0 a} J_1(k_0 a) - \cancelto{0}{J_0(k_0 a)}$$

$$-(k-k_0)a \, J_1(k_0 a) + \frac{(ka)^2}{2} \frac{J_1(k_0 a)}{k_0 a} \frac{1}{2}\sum_p \left(\epsilon_p^2 + \bar{\epsilon}_p^2\right)$$

$$+ \frac{(ka)^2}{2} \sum_m \frac{J_1(k_0 a)\, J_m'(k_0 a)}{J_m(k_0 a)} \left(\epsilon_m^2 + \bar{\epsilon}_m^2\right) = 0$$

$$\frac{(k-k_0)a}{k_0 a} = \frac{1}{2} \sum_{p=2}^{\infty} \left[\frac{1}{2} + \frac{k_0 a \, J_m'(k_0 a)}{J_m(k_0 a)}\right] \left(\epsilon_p^2 + \bar{\epsilon}_p^2\right)$$

$$\boxed{\frac{\delta \omega_{0n}}{\omega_{0n}} = \frac{1}{2} \sum_{p=2}^{\infty} \left[\frac{1}{2} + \alpha_{0n} \frac{J_m'(\alpha_{0n})}{J_m(\alpha_{0n})}\right] \left(\epsilon_p^2 + \bar{\epsilon}_p^2\right)}$$

Can show that $\left[ \frac{1}{2} + \frac{\alpha_{mn} J_m'}{J_m} \right] > 0$

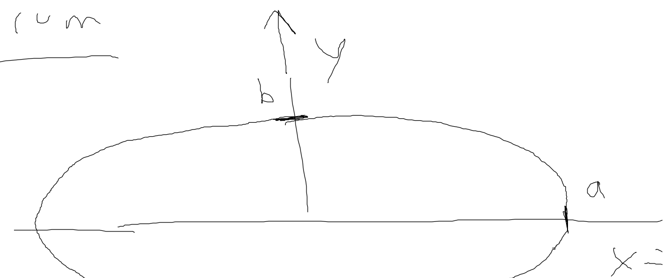
Therefore $\delta\omega > 0$

Change in ground state frequency is positive

$\Rightarrow$ Circle has lowest frequency of any nearly circular drum

<u>Example: Elliptical drum</u>

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$




$x = r\cos\phi \qquad y = r\sin\phi$

$b^2 = a^2(1-\epsilon^2) \qquad \epsilon$ = eccentricity

Note $\epsilon \neq \epsilon_p$. However now calculate $\epsilon_p$ given $\epsilon$!

$$\frac{b^2}{a^2}x^2 + y^2 = b^2 = r^2 - \epsilon^2 r^2 \cos^2\phi$$

$$r = \frac{b}{(1-\epsilon^2\cos^2\phi)^{1/2}} \simeq b\left(1 + \frac{\epsilon^2}{2}\cos^2\phi\right)$$

$$r \simeq b\left(1 + \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4}\cos 2\phi\right)$$

Area of ellipse $A = \pi a b$

$$\bar{a} = (ab)^{1/2} = b(1+\epsilon^2)^{1/4} \simeq b(1+\epsilon^2/4)$$

$$r \simeq \bar{a}\left(1 + \frac{\epsilon^2}{4}\cos 2\phi\right)$$

$$\Rightarrow \boxed{\epsilon_2 = \frac{\epsilon^2}{4} \quad \text{all other } \epsilon_p, \bar{\epsilon}_p = 0}$$

$$\frac{\delta\omega_{01}}{\omega_{01}} = \frac{1}{2}\left[\frac{1}{2} + \frac{J_2'(\alpha_{01})\,\alpha_{01}}{J_2(\alpha_{01})}\right]\epsilon_2^2$$

$$\frac{\delta\omega_{01}}{\omega_{01}} = \frac{\epsilon^4}{32}(1.3916)$$

$\alpha_{01} = 2.4048$
$J_2(\alpha_{01}) = .432$
$J_2' = .160$

$$\boxed{\frac{\delta\omega_{01}}{\omega_{01}} = 0.0435\ \epsilon^4} + O(\epsilon^6)$$

Example $\epsilon = \frac{1}{\sqrt{2}} \Rightarrow b = \frac{a}{2}$

$$\frac{\delta\omega_{01}}{\omega_{01}} \simeq 0.0109$$ 1% higher frequency