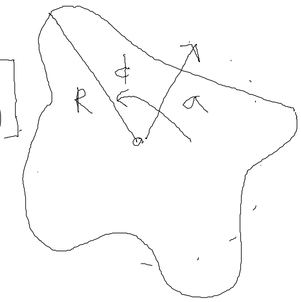
1/27/00

Lec 30 Nearly Circular Membrane

Expand b.c. in fourie sienes

$$R(\phi) = a\left[1 + \sum_{p=1}^{\infty}\left(\epsilon_p \cos p\phi + \bar{\epsilon}_p \sin p\phi\right)\right]$$



(A) $$[\nabla^2 + k^2]\,\rho(r,\phi) = 0$$

(B) $$\rho(r=R(\phi),\phi) = 0$$

Chose origin of circle

$$\epsilon_1 = \bar{\epsilon}_1 = 0 \qquad \langle R \rangle = a$$

Area of membrane

$$A = \int_0^{2\pi} d\phi \int_0^{R(\phi)} r\,dr = \int_0^{2\pi} d\phi\, \frac{R^2(\phi)}{2}$$

$$= \frac{a^2}{2}\int_0^{2\pi} d\phi \left[1 + 2\sum_{p=2}^{\infty}\left(\epsilon_p \cos p\phi + \bar{\epsilon}_p \sin p\phi\right) + O(\epsilon^2)\right]$$

$$A = \pi a^2 + O(\epsilon^2)$$

Expand general solution of (A) in solutions for a circular membrane. These don't satisfy our b.c. but are a complete set

$$\rho(r,\phi) = \sum_{m=0}^{\infty} J_m(kr)\left(A_m \cos m\phi + B_m \sin m\phi\right)$$

Chose k, A_m B_m to satisfy (B)

Expect $A_0 = 1$ convient normalization

$$A_m, B_m = O(\epsilon) \qquad m \geq 1$$

Expand

$$p\,(R(\phi),\phi) = 0$$

To first order in ϵ.

$$J_0\left\{ka\left[1+\sum_{p=2}^{\infty}\left(\epsilon_p\cos p\phi + \bar{\epsilon}_p \sin p\phi\right)\right]\right\}$$
$$+ \sum_{m=1}^{\infty} J_m(kR(\phi))\,\left(A_m\cos m\phi + B_m \sin m\phi\right) = 0$$

To first order $J_0(x) \simeq J_0(x_0) + (x-x_0)J_0'(x_0)$
$J_m(kR) \simeq J_m(ka)$ because A_m, B_m all ready order ϵ

$$J_0(ka) + ka\,J_0'(ka)\sum_{p=2}^{\infty}\left(\epsilon_p\cos p\phi + \bar{\epsilon}_p\sin p\phi\right)$$
$$+\sum_{m=1}^{\infty} J_m(ka)\left(A_m\cos m\phi + B_m\sin m\phi\right) = 0$$

True for all ϕ. $\Rightarrow$ Must be true for each power of $\cos m\phi$, $\sin m\phi$

$m=0$: $\boxed{J_0(ka) = 0}$

$m=1$: $J_1(ka)\,A_1 = 0 \Rightarrow$ $\boxed{\begin{array}{l}A_1 = 0\\ B_1 = 0\end{array}}$
$J_1(ka)\,B_1 = 0 \Rightarrow$

$m \geq 2$: $ka\,J_0'(ka)\,\epsilon_m = -J_m(ka)\,A_m$
$ka\,J_0'(ka)\,\bar{\epsilon}_m = -J_m(ka)\,B_m$

Note $J_0' = -J_1(ka)$

$$\boxed{\begin{array}{ll} A_m = \epsilon_m\,ka\,J_1(ka)/J_m(ka) & m \geq 2 \\ B_m = \bar{\epsilon}_m\,ka\,J_1(ka)/J_m(ka) & \end{array}}$$

From $m=0$ equation

$$J_0(ka) = 0 \Rightarrow k = \alpha_{0,n}/a$$

$$\omega_{0n} = \alpha_{0,n}\, c/a + O(\epsilon^2)$$

For nearly circularly symmetric modes. Thus ω is unchanged to order ϵ as shape is distorted from a circle

Lowest mode $\alpha_{0,1} = 2.4048$

$$\frac{\omega_{01}}{c} = \frac{2.4048}{a} = \frac{2.4048\ \pi^{1/2}}{A^{1/2}} = \boxed{\frac{4.2624}{A^{1/2}}}$$

Compare this to a square

$$\frac{\omega_{11}}{c}\Big|_{\text{square}} = \frac{(2\pi^2)^{1/2}}{a} = \frac{(2\pi^2)^{1/2}}{A^{1/2}} = \boxed{\frac{4.4429}{A^{1/2}}}$$

The frequency for a square is only 4% higher than for a circle.

Note a square is not a small perturbation on a circle. Nevertheless the result is consistent with the frequency being unchanged to order ϵ.

If one expands $\rho(r=R(\phi), \phi) = 0$ to second order in ϵ one can show

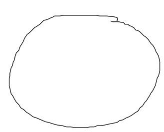
$$\frac{\delta\omega_{0n}}{\omega_n} = \frac{1}{2}\sum_{p=2}^{\infty}\left[1 + \frac{\alpha_{0,n} J_p'(\alpha_{0,n})}{J_p(\alpha_{0,n})}\right](\epsilon_p^2 + \bar{\epsilon}_p^2) \geq 0$$

Since this is a positive function of $\epsilon_p^2, \bar{\epsilon}_p^2$ we conclude

$\Rightarrow$ A circular membrane has the lowest frequency of any shaped membrane of a given area.

It is consistent with the square being 4% higher than a circle.

Clearly in the limit of a long very narrow rectangle the frequency will be much higher than for a circle.



Circle

Long rectangle of same area. Need lots of curvature in y direction