Lec 38 Nearly Circular Membrane

Expand b.c. in Fourier series

\[ R(\psi) = \alpha \left[ 1 + \sum_{p=1}^{\infty} \left( \varepsilon_p \cos p\psi + \varepsilon_p \sin p\psi \right) \right] \]

\[ \Omega \left[ \nabla^2 + k^2 \right] \rho (r, \psi) = 0 \]

\[ \rho (r = R(\psi), \psi) = 0 \]

Chose origin of circle

\[ \varepsilon_i = \varepsilon_{ij} = 0 \quad < R > = a \]

Area of Membrane

\[ A = \frac{2\pi}{\int_0^{2\pi} \frac{R^2(\psi)}{2} \cos \psi \, d\psi} \]

\[ = \frac{2\pi}{\int_0^{2\pi} \left( 1 + 2 \sum_{p=1}^{\infty} (\varepsilon_p \cos p\psi + \varepsilon_p \sin p\psi) + O(\varepsilon^2) \right) \frac{R^2(\psi)}{2} \cos \psi \, d\psi} \]

\[ A = \pi a^2 + O(\varepsilon^2) \]

Expand general solution of (A) in solution for a circular membrane. These don't satisfy our b.c. but are a complete set

\[ \rho (r, \psi) = \sum_{M=0}^{\infty} J_1(kr) (A_m \cos m\psi + B_m \sin m\psi) \]

Chose \( k, A_m, B_m \) to satisfy (B)

Expect \( A_0 = 1 \) because normalizer

\[ A_m, B_m = O(\varepsilon) \quad m \geq 1 \]
Expand

\[ P \left( R(q) \right) = 0 \]

To first order in \( \epsilon \),

\[ J_0(ka) \left[ 1 + \sum_{p=0}^{\infty} \left( \epsilon \cos pf + \tilde{\epsilon} \sin pf \right) \right] \]

\[ + \sum_{m=1}^{\infty} J_m(ka) \left( A_m \cos mf + B_m \sin mf \right) = 0 \]

To first order in \( \epsilon \),

\( J_0(x) = J_0(x_0) + (x-x_0) J_0'(x_0) \)

\( J_m(ka) \approx J_m(ka) \) because \( A_m \) and \( B_m \) are already zero.

\[ J_0(ka) + ka J_0'(ka) \sum_{p=0}^{\infty} \left( \epsilon \cos pf + \tilde{\epsilon} \sin pf \right) \]

\[ + \sum_{m=1}^{\infty} J_m(ka) \left( A_m \cos mf + B_m \sin mf \right) = 0 \]

True for \( \epsilon \) all \( f \) \( \Rightarrow \) Must be true for each power of \( \cos mf \) and \( \sin mf \)

\( m = 0 : \) \[ J_0(ka) = 0 \]

\( m = 1 : \) \[ J_1(ka) A_1 = 0 \]

\[ J_1'(ka) B_1 = 0 \] \( \Rightarrow \)

\[ A_1 = 0 \]

\[ B_1 = 0 \]

\( m \geq 2 : \) \[ \begin{aligned} ka J_0'(ka) \epsilon_m &= -J_m(ka) A_m \\ ka J_0'(ka) \tilde{\epsilon}_m &= -J_m(ka) B_m \end{aligned} \]

\( \text{Note:} \) \[ J_0' = -J_1(ka) \]

\[ A_m = \epsilon_m \frac{ka J_1(ka)}{J_m(ka)} \quad m \geq 2 \]

\[ B_m = \tilde{\epsilon}_m \frac{ka J_1(ka)}{J_m(ka)} \]
From m=0 equation
\[ J_0(k\alpha) = 0 \Rightarrow k = \alpha_0, n / a \]
\[ \omega_n = \alpha_0 \frac{c}{a} + O(\alpha^2) \]

For nearly circularly symmetric modes, thus \( \omega_n \) is unchanged to order \( \epsilon \) as shape is distorted from a circle.

Lowest mode \( \alpha_0, 1 = 2.4048 \)
\[ \frac{\omega_1}{c} = \frac{2.4048}{\alpha} = \frac{2.4048}{\pi^2} = \frac{4.2624}{A^{1/2}} \]

Compare this to a square
\[ \frac{\omega_{11}}{c} = \frac{(2\pi^2)^{1/2}}{\alpha} = \frac{(2\pi)^{1/2}}{A^{1/2}} = \frac{4.4449}{A^{1/2}} \]

The frequency for a square is only 4% higher than for a circle.

Note a square is not a small perturbation on a circle. Nevertheless the result is consistent with the frequency being unchanged to order \( \epsilon \).

IF one expands \( \omega_n \) in \( \omega \) to second order in \( \epsilon \)
\[ \frac{\Delta \omega_n}{\omega_n} = \frac{1}{2} \sum_{\alpha_0, n} \frac{\alpha_0^2}{\omega_n} \left( 1 + \frac{\omega_n J_0'(\alpha_0, n)}{J_0(\alpha_0, n)} \right) \left( \epsilon^2 + \epsilon^2 \right) \geq 0 \]
Since this is a positive function of $\varepsilon^2_1/\varepsilon^2_p$, we conclude

A circular membrane has the lowest frequency of any shaped membrane of a given area.

It is consistent with the square being 4.5% higher than a circle.

Clearly in the limit of a long, very narrow rectangle the frequency will be much higher than for a circle.

![Diagram]

Long rectangle of same area. Need more of curvature in y direction.