11/20/06

Lec 37 Membranes

Start reading chapter 8
No homework this week

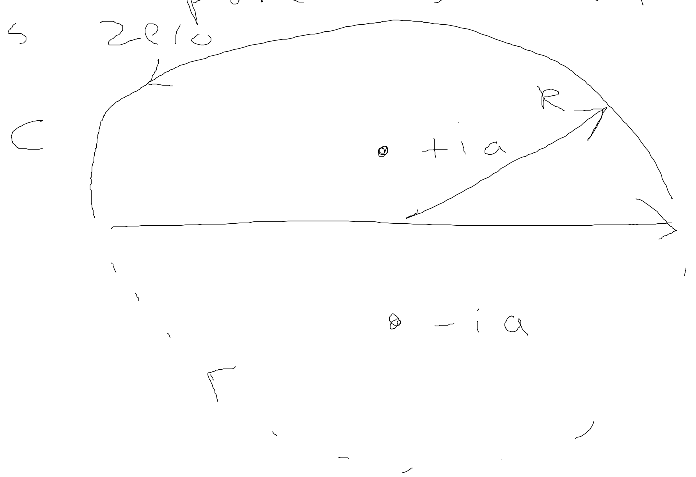
Last time contour integration

$$\oint_C f(z)\, dz = 2\pi i \sum \text{Residues}$$

Example of contour integration

$$I = \int_{-\infty}^{\infty} dx \, \frac{e^{ikx}}{x^2 + a^2}$$

Close contour in either upper or lower half plane so contribution of semicircle is zero



$$e^{ikz} = e^{ik \operatorname{Re} z} e^{-k \operatorname{Im} z} \qquad \text{want } \operatorname{Im} z > 0$$

Chose upper half plane. Contribution of semicircle is zero as $R \to \infty$

$$I = \oint_C dz \, \frac{e^{ikz}}{z^2 + a^2}$$

Poles at $z = \pm ia$

Only pole at $z = +ia$ is inside contour
Calculate residue at $z = +ia$

$$\frac{1}{z^2+a^2} = \frac{1}{z+ia}\ \frac{1}{z-ia}$$

$$\simeq \frac{1}{2ia}\ \frac{1}{z-ia} \quad \text{as} \quad z \to ia$$

$$I = 2\pi i \ \frac{1}{2ia}\ e^{ikia} = \boxed{\frac{\pi}{a} e^{-ka}}$$

2nd example

$$I = \int_{-\infty}^{\infty} dx \ \frac{\cos kx}{x^2+a^2}$$

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \ \frac{e^{ikx}}{x^2+a^2} + \frac{1}{2}\int_{-\infty}^{\infty} dx \ \frac{e^{-ikx}}{x^2+a^2}$$

$$= \frac{\pi}{2a} e^{-ka} + \frac{1}{2}\oint_{C_2} \frac{e^{-ikz}}{z^2+a^2}\, dz$$

Chose C_2 for 2nd integral $e^{-ikz} = e^{-ik\,\mathrm{Re} z}\, e^{kI}$
Need $\mathrm{Im}\, z < 0$ for circular arc.

$$\frac{1}{z^2+a^2} = \frac{1}{z+ia}\ \frac{1}{z-ia} \simeq -\frac{1}{2ia}\ \frac{1}{z+ia}$$

$$\lim z \to -ia$$

However there is an extra minus sign for going around contour in clockwise direction

C_a $\qquad\qquad$ C_b

$$\oint_{C_a} f(z)\,dz = -\oint_{C_b} f(z)\,dz$$

$$\oint_{C_2} dz \left[\frac{e^{-ikz}}{z^2+a^2}\right] = (2\pi i)(-1)\frac{e^{-ik(-ia)}}{(-2ia)}$$

$$= \frac{\pi}{a}e^{-ka}$$

So $$\int_{-\infty}^{\infty} dx\, \frac{\cos kx}{x^2+a^2} = \frac{1}{2}\left[\frac{\pi}{a}e^{-ka} + \frac{\pi}{a}e^{-ka}\right]$$

$$= \frac{\pi}{a}e^{-ka}$$

Membranes

$u(x,t) \longrightarrow u(x,y,t)$

Displacement of membrane in z direction at point (x,y)

$$\mathcal{T} = \frac{1}{2}\sigma(x,y)\left(\frac{\partial u}{\partial t}\right)^2$$ Kinetic E Density

Total mass $M = \int \sigma(x,y)\,dx\,dy$

σ = Mass per unit area

$$\mathcal{V} = \frac{1}{2}\tau\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

= force/length
τ = surface tension
= energy/area

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\tau}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$L = \int dx\, dy\, \mathcal{L}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial \frac{\partial u}{\partial y}} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\sigma(\vec{x})\frac{\partial^2 u}{\partial t^2} = \vec{\nabla}\cdot \tau(\vec{x})\,\vec{\nabla} u$$

Uniform membrane σ = const, τ = const

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$$

$$c^2 = \tau/\sigma$$

Normal modes

$$u(x,t) = \rho(\vec{x})\cos(\omega t + \phi)$$

$$\left[\nabla^2 + k^2\right]\rho = 0 \qquad k = \omega/c$$

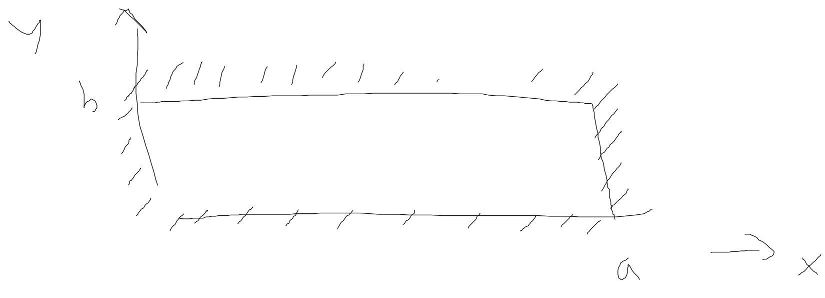
B. Conditions depend on geometry

Example Rectangular Membrane

$$\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + k^2\rho = 0$$

$$\rho(x,y) = X(x)\,Y(y)$$

B.C. Fixed edges $X(0) = X(a) = 0$
$Y(0) = Y(b) = 0$



$$\rho_{mn} = \left(\frac{2}{a}\right)^{1/2} \left(\frac{2}{b}\right)^{1/2} \sin(k_x x) \sin k_y y$$

$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{a} \qquad m, n = 1, 2, 3, \ldots$$

$$\omega^2_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Circular Membrane (Drum)

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

$$\left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + k^2\right] \rho(r, \phi) = 0$$

$$\rho = R(r)\, \Phi(\phi)$$

B.C. $\Phi(\phi + 2\pi) = \Phi(\phi)$ $\qquad \Phi_m = (2\pi)^{-1/2} e^{im\phi}$

$m = 0, \pm 1, \pm 2, \pm \ldots$

$R(a) = 0$

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m^2 \Phi$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}R(r)\right) - \frac{m^2}{r^2}R(r) + k^2 R(r) = 0$$

Put into Sturm - Liouville form

$$-\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{m^2}{r}R = k^2 r R(r)$$

Compare with original

$$-\frac{d}{dr}\tau\frac{dR}{dr} + V R = \omega^2 \sigma R$$

$$\tau = r, \quad \sigma = r \quad V = \frac{m^2}{r}$$

Solution $R = J_m(kr)$ or $N_m(kr)$

Bessel Neumann

functions. But N_m is singular at $r \to 0$
hence

$$R = C_m J_m(kr)$$

B.C. $J_m(ka) = 0 \Rightarrow k_{mn} = \frac{\alpha_{mn}}{a}$

α_{mn} is n^{th} zero of m^{th} Bessel func.

Example $\alpha_{01} = 2.4048\ldots$

From Sturm - Liouville theory $\int_0^a dr\, J_m(k_{mn}r)\, r\, J_m(k_{mn'}r) = 0 \quad n \neq n'$

Normalize

$$\int_0^a r\,dr\, J_m\left(\alpha_{mn}\frac{r}{a}\right)^2 = \frac{a^2}{2}\left[J_{m+1}(\alpha_{m,n})\right]^2$$

$$\boxed{\rho_{mn} = \frac{1}{\sqrt{\frac{1}{2}}\, a\, |J_{m+1}(\alpha_{m,n})|} J_m\left(\alpha_{m,n}\frac{r}{a}\right)\frac{e^{im\phi}}{\sqrt{2\pi}} \quad \begin{array}{l} m = 0, \pm 1, \pm 2, \pm 3 \ldots, \pm\infty \\ n = 1, 2, \ldots, \infty \end{array}}$$

$$\int_0^a \underset{\uparrow \sigma}{r}\,dr \int_0^{2\pi} d\phi\, \rho^*_{mn}\, \rho_{m'n'} = \delta_{mm'}\delta_{nn'}$$

General solution

$$u(r,\phi,t) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \rho_{mn}(r,\phi)\left[\underbrace{a_{mn}\cos\omega_{mn}t + b_{mn}\sin\omega_{mn}t}_{C_{mn}\cos(\omega_{mn}t + \phi_{mn})}\right]$$