Lec 36

Complex Integration

11/17/00

See appendix A of F+W

Show two forms of Green's func. are equivalent

$$G_\omega(x,y) = \sum_n \frac{\rho_n(x)\ \rho_n(y)}{\omega_n^2 - \omega^2}$$

$$= -u_1(x_<)\ u_2(x_>)\ /\ C$$

For a uniform string

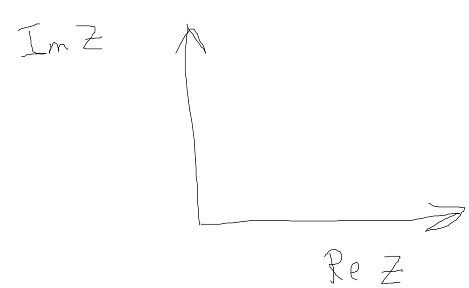
$$G_0 = \frac{2}{\sigma l} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{l}\right)\ \sin\frac{n\pi y}{l}}{\omega_n^2 - \omega^2}, \qquad \omega_n = \frac{n\pi c}{l}$$

$$G_1 = \frac{\sin(\omega x_</c)\ \sin\frac{\omega}{c}(l - x_>)}{\frac{\tau}{c}\,\omega\ \sin\left(\frac{\omega l}{c}\right)}, \qquad c = \sqrt{\frac{\tau}{\sigma}}$$

Want to show $G_0 = G_1$

Complex integration $\Rightarrow$ G_0 into complex plain

$Z = x + iy$

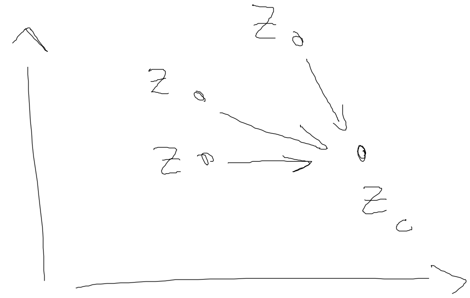


Analytic Functions

A function is analytic in a region Γ if the derivative exists and is unique (for every point z in Γ)

$$f'(z_0) = \lim_{z_0 \to z} \frac{f(z) - f(z_0)}{z - z_0}$$

f' should be independent of direction how $z \to z_0$

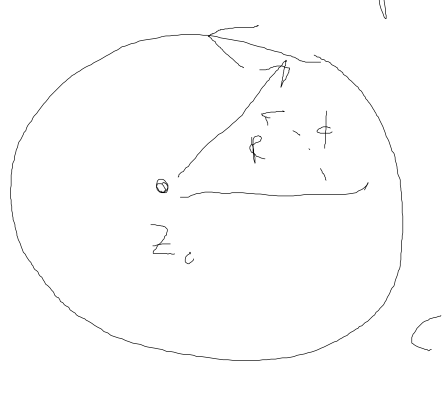


An analytic function can be expanded in a Taylor series

$$f(z) \approx f(z_0) + (z - z_0)\, f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2} + \dots$$

Consider a contour integral around a circle of some power $(z - z_0)^n$

$$z - z_0 = Re^{i\phi}$$



$$dz = R i e^{i\phi} d\phi$$

$$\oint_C f(z)\, dz = \int_0^{2\pi} f(z)\, iR\, d\phi\, e^{i\phi}$$

To go around contour C R stays fixed and ϕ goes from 0 to 2π.

Let $f = (z - z_0)^n = R^n e^{in\phi}$

$$\oint_C f(z)\, dz = i R^{n+1} \int_0^{2\pi} e^{i(n+1)\phi}\, d\phi = 0$$

For any $n \geq 0$

Can expand any analytic function in a power series. All terms in power series integrate to zero.

$$\Rightarrow \boxed{\oint_C f(z)\, dz = 0}$$

For any analytic function f.

We have proved this for a circular contour C of any radius R. Cauchy's theorem

$$\oint_C f(z)\, dz = 0 \quad \text{for any closed contour } C.$$

Nonanalytic functions can have poles

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2}$$

$$+ \dots \frac{b_k}{(z-z_0)^k}$$

If $b_n = 0$ for $n > k$ than f has a k^{th} order pole at z_0

If $k \to \infty$ f has an essential singularity at z_0

$$\oint_{C_0} \frac{1}{z^n} dz = i R^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} d\phi$$

$$= 2\pi i \, \delta_{n,1}$$

only $\frac{1}{z-z_0}$ gives a nonzero contribution.

This contribution is independent of R.

Important result

$$\oint_C f(z)\, dz = 2\pi i \, b_1$$

b_1 is said to be the residue of f at $z_0 = R_1$

If a function has multipule poles inside C than

$$\boxed{\oint_C f(z)\, dz = 2\pi i \sum \text{Residues}}$$

Expand f about each pole location z_0 and determine coef of $\frac{1}{z-z_0}$ term.

This is residue. All other coef. do not matter

Example

$$G_0 = \frac{2}{\sigma l} \sum_n \frac{\sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right)}{\omega_n^2 - \omega^2}$$

has poles at $\omega = \pm \frac{n\pi c}{l}$

$$\frac{1}{\omega_n^2 - \omega^2} = \left[\frac{1}{\omega_n + \omega}\right]\left[\frac{1}{\omega_n - \omega}\right]$$

for $\omega \to \omega_n$

$$\approx -\frac{1}{2\omega_n} \frac{1}{\omega - \omega_n}$$

Thus residue of $\frac{1}{\omega_n^2 - \omega^2}$ at ω_n is $\frac{-1}{2\omega_n}$

$$G_0 \approx -\frac{1}{\sigma l} \sum_n \frac{R_n}{\omega - \omega_n} \qquad \text{for } \omega > 0$$

$$R_n = \frac{1}{[\text{illegible}]} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right)$$

Consider

$$G_1 = \frac{\sin(\omega x_</c)\ \sin\left(\frac{\omega}{c}\ l - x_>\right)}{\frac{T}{c}\ \omega\ \sin\left(\frac{\omega l}{c}\right)}$$

This also has poles at $\omega = \pm \frac{n\pi c}{l}$

$$\sin\left(\frac{\omega l}{c}\right) \approx (\omega - \omega_n)\ \frac{l}{c} \cos\left(\frac{n\pi c}{l}\left(\frac{l}{c}\right)\right)$$

$$\frac{1}{\sin\left(\frac{\omega l}{c}\right)} \approx \left[\frac{1}{\frac{l}{c}(-1)^n}\right] \frac{1}{\omega - \omega_n}$$

Interested in $\omega \to \omega_n$ so simply evaluate numerator with $\omega = \omega_n$

$$G_1 \approx \frac{\sin\left(\frac{\omega_n x_<}{c}\right)\ \sin(n\pi - \omega_n x_>/c)}{\frac{T}{c}\ \omega_n\ \frac{l}{c}\ (-1)^n\ (\omega - \omega_n)}$$

$$\sin(n\pi - \omega_n x_>/c) = (-1)^{n+1} \sin(\omega_n x_>/c)$$

$$G_1 \approx \frac{(-1)\ \sin\left(\frac{\omega_n x_<}{c}\right) \sin(\omega_n x_>/c)}{\frac{Tl}{c^2}\ \omega_n\ (\omega - \omega_n)}$$

$$= -\frac{1}{l\sigma}\ \frac{\sin\left(\frac{\omega_n x_<}{c}\right)\ \sin(\omega_n x_>/c)}{\omega_n\ (\omega - \omega_n)}$$

Thus G_1 has same residue as G_c at each pole $\omega \to \omega_n = \frac{n\pi c}{l}$