Lec 36
Complex Integration

See appendix A of F+W

Show two forms of Green’s func. are equivalent

\[ G(x, y) = \sum_{n} \frac{p_n(x) p_n(y)}{w_n^2 - w^2} \]

\[ = -u_1(x) u_2(x) / C \]

For a uniform string

\[ G_0 = \frac{2}{\alpha L} \sum_{n=1}^{\infty} \frac{\sin(\frac{m \pi x}{L}) \sin(\frac{n \pi y}{L})}{w_n^2 - w^2} \]

\[ C_n = \frac{\sin(w_n x/c) \sin(\frac{w_n (L-x)}{c})}{\frac{c}{u} w \sin(\frac{w_n L}{c})} \]

Want to show \( G_0 = G_1 \)

Complex integration \( \Rightarrow \) Go into complex plane

\[ Z = x + iy \]

Im Z

Re Z
Analytic Functions

A function is analytic in a region \( \mathcal{D} \) if the derivative exists and is unique (for every point \( z \) in \( \mathcal{D} \))

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]

It should be independent of direction, how \( z \to z_0 \).

An analytic function can be expanded in a Taylor series

\[
f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2} + \cdots
\]

Consider a contour integral around a circle of some power \((z - z_0)^n\)

\[
z - z_0 = \text{Re} i \theta
\]
\[ dz = R e^{i\phi} d\phi \]

\[ \oint_C F(z) \, dz = \int_0^{2\pi} F(R e^{i\phi}) R e^{i\phi} i R \, d\phi = 0 \]

To go around contour \( C \) \( R \) stays fixed and \( \phi \) goes from 0 to \( 2\pi \).

Let \( F = (z - z_0)^n = R^n e^{in\phi} \)

\[ \oint_C F(z) \, dz = i R^{n+1} \int_0^{2\pi} e^{in\phi} e^{i\phi} \, d\phi = 0 \]

For any \( n \geq 0 \)

Can expand any analytic function in a power series. All terms in power series integrate to zero.

\[ \Rightarrow \oint_C F(z) \, dz = 0 \]

For any analytic function \( f \).

We have proved this for a circular contour \( C \) of any radius \( R \). Cauchy's theorem \[ \oint_C F(z) \, dz = 0 \] for any closed contour \( C \).
Nonanalytic functions can have poles

\[ F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_k}{(z-z_0)^k} \]

If \( b_n = 0 \) for \( n \geq k \) then \( F \) has a \( k \)th order pole at \( z_0 \).

If \( k \to \infty \) \( F \) has an essential singularity at \( z_0 \).

\[ \oint_{C} \frac{1}{z^n} dz = i \pi \delta_{n,1} \]

It is only \( \frac{1}{z-z_0} \) that gives a nonzero contribution. This contribution is independent of \( R \).

Important result

\[ \oint_{C} F(z) \, dz = 2\pi i \cdot b_1 \]

\( b_1 \) is said to be the residue of \( F \) at \( z_0 = R \).
If a function has multiple poles inside \( C \), then
\[
\oint_C f(z) \, dz = 2\pi i \sum \text{Residues}
\]

Expand \( f \) about each pole location \( z_0 \) and determine each coefficient of \( \frac{1}{z - z_0} \) term.

This is residue. All other coefficients do not matter.

Example

\[
G_0 = \frac{2\pi}{\sin nN} \sum_{n} \frac{\sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{m\pi y}{L} \right)}{\omega_n^2 - \omega^2}
\]

has poles at \( \omega = \pm \frac{m\pi c}{L} \)

\[
\frac{1}{\omega_n^2 - \omega^2} = \frac{1}{\omega_n + \omega} \frac{1}{\omega_n - \omega}
\]

So, \( \omega \to \omega_n \)

\[
\sum_{n} \frac{1}{\omega_n} \to \frac{1}{2\omega_n} \frac{1}{\omega - \omega_n}
\]

Thus, residue of \( \frac{1}{\omega_n^2 - \omega^2} \) at \( \omega_n \) is \( -\frac{1}{2\omega_n} \)

\[
G_0 \approx \frac{-1}{6L} \sum_{n} \frac{R_n}{\omega_n - \omega_n}
\]

\[
R_n = \frac{1}{\sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{m\pi y}{L} \right)}
\]
Consider
\[ G_1 = \frac{\sin \left( \frac{\omega l}{c} \right) \sin \left( \frac{\omega}{c} l - x_2 \right)}{\frac{\omega}{c} \sin \left( \frac{\omega l}{c} \right)} \]

This also has poles at \( \omega = \pm \frac{n \pi c}{l} \)

\[ \sin \left( \frac{\omega l}{c} \right) = (\omega - \omega_n) \frac{l}{c} \cos \left( \frac{n \pi c}{l} \frac{l}{c} \right) \]

\[ \frac{1}{\sin \left( \frac{\omega l}{c} \right)} = \left[ \frac{1}{\frac{l}{c} (-1)^n} \right] \frac{1}{\omega - \omega_n} \]

Interested in \( \omega \to \omega_n \) so simply evaluate numerator with \( \omega = \omega_n \)

\[ G_1 = \frac{\sin \left( \frac{\omega_n l}{c} \right) \sin \left( \frac{n \pi - \omega_n x_2}{c} \right)}{\frac{\omega_n}{c} \frac{l}{c} (-1)^n (\omega - \omega_n)} \]

\[ \sin \left( \frac{n \pi - \omega_n x_2}{c} \right) = (-1)^{n+1} \sin \left( \frac{\omega_n x_2}{c} \right) \]

\[ G_1 = (-1) \frac{\sin \left( \frac{\omega_n l}{c} \right) \sin \left( \frac{\omega_n x_2}{c} \right)}{\frac{\omega_n}{c} (\omega - \omega_n)} \]

\[ = \frac{-1}{l} \frac{\sin \left( \frac{\omega_n l}{c} \right) \sin \left( \frac{\omega_n x_2}{c} \right)}{\omega_n (\omega - \omega_n)} \]

Thus \( G_1 \) has same residue as \( G_0 \) at each pole \( \omega \to \omega_n = \frac{n \pi c}{l} \).