11/15/00

Lecture 35 Perturbation Theory

Given exact solution of one problem

$$-\frac{d}{dx}\tau\frac{d}{dx}\rho_n + V\rho_n = \omega_n^2\sigma\rho_n$$

Find approx. solution to $V' = V + \epsilon U(x)\sigma(x)$ with $\epsilon << 1$

$$-\frac{d}{dx}\tau\frac{d}{dx}\rho + V\rho + \epsilon U\sigma\rho = \omega^2\sigma\rho$$

$$L_0\rho_n = \omega_n^2\sigma\rho_n$$

$$(L_0 + L_1)\rho = \omega^2\sigma\rho$$

$$L_0 \equiv -\frac{d}{dx}\tau\frac{d}{dx} + V \qquad L_1 = \epsilon U(x)\sigma(x)$$

$$[L_0 - \omega^2\sigma]\rho = -L_1\rho$$

Use Green's function from last time

$$\rho(x) = -\int_a^b dx\, G_\omega(x,y)L_1\rho(y)\,dy$$

$$G_\omega = \sum_n \frac{\rho_n(x)\rho_n(y)}{\omega_n^2 - \omega^2}$$

$$\rho(x) = \sum_{m=1}^{\infty} \rho_m(x) \frac{1}{\omega^2 - \omega_m^2} \int_a^b \rho_m(y) L_1 \rho(y)\, dy$$

Integral equation, unknown ρ on both sides
Above of form

$$\rho(x) = \sum_{m=1}^{\infty} c_m \rho_m(x)$$

Chose normalization for ρ

$$\langle \rho_n | \rho \rangle \equiv \int_a^b dx\, \rho_n(x)\, \sigma(x)\, \rho(x) = 1$$

$$1 = \sum_m \langle \rho_n | \rho_m \rangle \frac{1}{\omega^2 - \omega_m^2} \int_a^b \rho_m(y) L_1 \rho(y)\, dy$$

Project out multiply by $\rho_n(x)\, \sigma(x)$ and integrate

(1) $$\boxed{\omega^2 - \omega_n^2 = \int_a^b \rho_n(y) L_1 \rho(y)\, dy}$$

Gives exact frequency ω^2 also

(2) $$\boxed{\rho(x) = \rho_n(x) + \sum_{q \neq n} \rho_q(x) \frac{1}{\omega^2 - \omega_q^2} \int_a^b \rho_q(y) L_1 \rho(y)\, dy}$$

Can solve these equations by iteration to express $\rho(x)$, ω^2 as power sieres in ϵ

If $\epsilon << 1$ start with $\rho \approx \rho_n$

(3) $$\rho(x) \approx \rho_n(x) + \epsilon \sum_{q \neq n} \rho_q \frac{1}{\omega_n^2 - \omega_q^2} \langle q|v|n\rangle + O(\epsilon^2)$$

Used $\omega^2 \approx \omega_n^2$ and

$$\langle q|v|n\rangle \equiv \int_a^b \rho_q(y)\, v(y)\, \rho_n(y)\, \sigma(y)\, dy$$

From (1) plug in (3)

$$\omega^2 - \omega_n^2 \approx \epsilon \langle n|v|n\rangle + \epsilon^2 \sum_{q \neq n} \langle n|v|q\rangle \frac{1}{\omega_n^2 - \omega_q^2} \langle q|v|n\rangle + O(\epsilon^3)$$

Note wave function to order ϵ gives ω^2 to order ϵ^2. Remember variational functional – error in ω^2 was second order in error in wave func.

Example Mass point on String

$$\sigma(x) = \sigma\left[1 + \frac{m}{\sigma}\delta\left(x - \frac{l}{2}\right)\right]$$

$$\sigma(x)\frac{\partial^2 u}{\partial t^2} = \tau\frac{\partial^2 u}{\partial x^2}$$

$$u = \rho(x)\cos(\omega t + c)$$

$$\tau\frac{d^2\rho}{dx^2} = -\omega^2\sigma(x)\rho$$

beat it into our form

$$-\tau\frac{d^2\rho}{dx^2} - \omega^2\frac{m}{\sigma}\delta\left(x - \frac{l}{2}\right)\sigma\rho = \omega^2\sigma\rho$$

$$\boxed{\epsilon\nu = -\omega^2\frac{m}{\sigma}\delta\left(x - \frac{l}{2}\right)}$$

$$-\tau\frac{d^2\rho_n}{dx^2} = \omega_n^2\sigma\rho_n$$

$$\rho_n = \left(\frac{2}{\sigma l}\right)^{1/2}\sin\frac{n\pi x}{l}, \qquad \omega_n = \frac{n\pi c}{l}$$

$$c = \sqrt{\tau/\sigma}$$

Assume $\dfrac{m}{M} = \dfrac{m}{\sigma l} \ll 1$

$$\omega^2 - \omega_n^2 = -\omega^2 \frac{m}{\sigma} \int_0^l \rho_n(x)\, \delta\left(x - \frac{l}{2}\right) \rho_n(x)\, \sigma\, dx$$

$\frac{m}{\sigma}$ is small so on RHS $\omega^2 \to \omega_n^2$

$$\omega^2 - \omega_n^2 \simeq -\omega_n^2 \frac{2m}{\sigma l} \sin^2\left(\frac{n\pi}{2}\right)$$

$$\omega = \omega_n + \delta\omega_n$$

$$\frac{\delta\omega_n}{\omega_n} = \begin{cases} -m/M & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

If one expands exact answer

$$\frac{1}{\xi} \cot \xi = \frac{m}{M}$$

$$\xi = \frac{\omega l}{2c} = \xi_n + \delta\xi_n$$

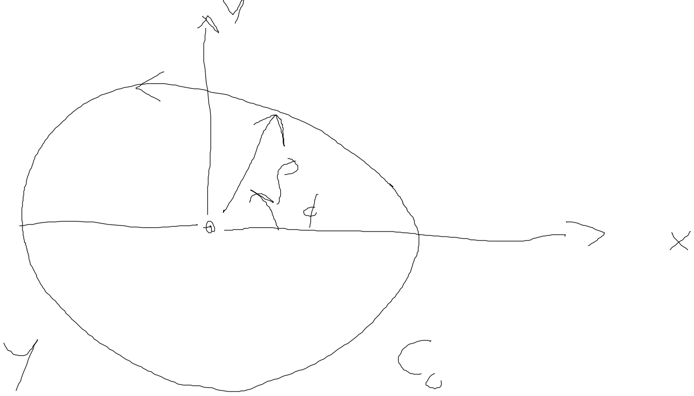
to order m/M find $\frac{\delta\xi_n}{\xi_n} = \begin{cases} -\frac{m}{M} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$$\frac{1}{\xi_n} \cot \xi_n = 0$$

$$\Rightarrow \xi_n = \frac{n}{2}\pi$$

Appendix A Complex analysis and Cauchy's theorem

Consider contour integration in complex plane



$z = x + iy$

$$z = \rho e^{i\phi}$$

$$dz = i\, d\phi\, \rho e^{i\phi}$$

Integrate around a closed circle

$$\oint_{C_0} z^n\, dz = i \rho^{n+1} \int_0^{2\pi} e^{i(n+1)\phi}\, d\phi = 0$$

$$\oint_{C_0} \frac{dz}{z^n} = i \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi}\, d\phi = 2\pi i\, \delta_{n,1}$$

The only nonzero integral is

$$\oint_{C_0} \frac{dz}{z} = 2\pi i$$

Analytic Function:

Can expand an analytic function in a Taylor series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

and $\oint_{C_0} f(z) = 0$

because all powers $\oint_{C_0} (z-z_0)^n dz = 0$

Cauchy's theorem: for an analytic function

$$\oint_C f(z)\, dz = 0$$

where C is any closed curve.