11/13/60

Lecture 34 Green's Functions

Driven string $\quad u(x,t) = \mathrm{Re}\, u(x)\, e^{-i\omega t}$

$$\sigma \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \tau \frac{\partial u}{\partial x} - vu + \mathrm{Re}\, \sigma f e^{-i\omega t}$$

$$u(x) = \int_a^b G_\omega(x,y')\, \sigma(y')\, f(y')\, dy'$$

if $\quad \sigma(y') f(y') = \delta(y-y')$

$$u(x) = G_\omega(x,y)$$

Green's function gives response to δ func. external force. ω is external freq. in general different from ω_n

Spectral expansion

$$G_\omega(x,y) = \sum_{n=1}^{\infty} \frac{\rho_n(x)\, \rho_n(y)}{\omega_n^2 - \omega^2}$$

$$[L_0 - \omega^2 \sigma]\, G_\omega(x,y) = \delta(x-y)$$

$$L_0 \equiv -\frac{d}{dx} \tau \frac{d}{dx} + V(x)$$

Note $\quad [L_0 - \omega^2\sigma]\, \rho_n(x) = [\omega_n^2 - \omega^2]\, \sigma \rho_n(x)$

because $[L_0 - \omega_n^2 \sigma]\rho_n = 0$

$$[L_0 - \omega^2\sigma]\, G_\omega(x,y) = \sum_{n=1}^{\infty} \rho_n(x)\,\sigma(x)\,\rho_n(y)$$

$$= \delta(x-y)$$ because ρ_n form complete set.

2nd Solution for Green's function

$$(L_0 - \omega^2\sigma)\, G_\omega^<(x,y) = 0 \qquad a \le x < y$$

$$G_\omega^<(x,y) = A(y)\, U_1(x)$$

U_1 satisfies b.c. at $x = a$

$$(L_0 - \omega^2\sigma)\, G_\omega^>(x,y) = 0 \qquad y < x \le b$$

$$G_\omega^>(y,y) = B(y)\, U_2(x)$$

U_2 satisfies b.c. at $x = b$

Match two at $x = y$

$$\int_{y-\epsilon}^{y+\epsilon} dx \left[-\frac{d}{dx}\tau\frac{d}{dx}G + VG - \omega^2\sigma G \right] = \int dx\, \delta(x-y)$$

$$-\tau \frac{d}{dx} G \Big|_{x=y-\epsilon}^{x=y+\epsilon} = 1$$

$$-\frac{1}{\tau(y)} = \frac{d}{dx} G^{>}_{\omega}(x=y+\epsilon, y) - \frac{d}{dx} G^{<}_{\omega}(x=y-\epsilon, y) \qquad \text{(A)}$$

On physical grounds $G_\omega(x,y)$ is a physical response to a δ func. therefore $G_\omega(x,y)$ is cont. at $x=y$.

$$G^{>}_{\omega}(x=y, y) = G^{<}_{\omega}(x=y, y) \qquad \text{(B)}$$

Solve (A), (B) for $A(y)$, $B(y)$

$$A = -\frac{u_2(y)}{\tau(y)\, W[u_1(y), u_2(y)]}, \quad G^{<}_{\omega} = A \cdot u_1(x)$$

$$B = -\frac{u_1(y)}{\tau(y)\, W[u_1(y), u_2(y)]}, \quad G^{>}_{\omega} = B\, u_2(x)$$

W = wronskian $\equiv u_1(y)\, u_2'(y) - u_2(y)\, u_1'(y)$

<u>Proof</u> $\tau(y)\, W[u_1(y), u_2(y)] = C$ independant of y

$$[L_0 - \omega^2 \sigma]\, u_1(x) = 0$$

$$[L_0 - \omega^2 \sigma]\, u_2(x) = 0$$

multiply first eq. by u_2 and subtract u_1 times second eq.

$$-u_2(x)\frac{d}{dx}\tau\frac{du_1}{dx} + u_1\frac{d}{dx}\tau\frac{d}{dx}u_2 = 0$$

or

$$\frac{d}{dx}\left\{\tau(x)\left[u_1(x)\frac{d}{dx}u_2 - u_2(x)\frac{d}{dx}u_1\right]\right\} = 0$$

so $\tau(u_1u_2' - u_2u_1') = C$

$$G_\omega(x,y) = A\,u_1(x) \qquad x<y$$

$$\boxed{G_\omega(x,y) = -\frac{u_2(y)\,u_1(x)}{C}} \qquad x<y$$

$$G_\omega(x,y) = B\,u_2(x)$$
$$= -\frac{u_1(y)\,u_2(x)}{C} \qquad y<x$$

$$\boxed{G_\omega(x,y) = -\frac{u_1(x_<)\,u_2(x_>)}{C}}$$

u_1 satisfies b.c. at a $\qquad x_< = \min\{x,y\}$

u_2 " " at b

Example: Uniform string fixed ends

$$\rho_n = \left(\frac{2}{\sigma l}\right)^{1/2} \sin\left(\frac{n\pi x}{l}\right) \qquad n = 1, 2 \ldots \infty$$

$$\omega_n = \frac{c\, n\pi}{l} \qquad c = \sqrt{\frac{\tau}{\sigma}}$$

$$\boxed{G_\omega(x,y) = \frac{2}{\sigma l} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right)}{\left(\frac{c n \pi}{l}\right)^2 - \omega^2}}$$

$$L_0 = -\tau \frac{d^2}{dx^2}$$

$$\left[L_0 - \omega^2 \sigma\right] G_\omega = \frac{2\sigma}{\sigma l} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right)$$

$$= \delta(x-y) \qquad \text{completeness}$$

2nd Solution

$$\left[L_0 - \omega^2 \sigma\right] U_1(x) = 0$$

$$U_1 = \sin \frac{\omega x}{c} \qquad U_1(0) = 0$$

$$\left[L_0 - \omega^2 \sigma\right] U_2(x) = 0$$

$$U_2 = \sin\left[\frac{\omega}{c}(l - x)\right] \qquad U_2(l) = 0$$

$$C = \tau\, W[u_1, u_2]$$

$$= \tau \left[\sin\left(\frac{\omega x}{c}\right) \left(-\frac{\omega}{c}\right) \cos\left[\frac{\omega}{c}(l - x)\right] - \sin\left[\frac{\omega}{c}(l - x)\right] \cos\left(\frac{\omega x}{c}\right) \left(\frac{\omega}{c}\right) \right]$$

$$= -\tau \frac{\omega}{c} \sin\frac{\omega l}{c}$$

$$G_\omega(x, y) = -\frac{u_1(x_<)\, u_2(x_>)}{C}$$

$$\boxed{G_\omega(x, y) = \frac{\sin\left(\frac{\omega}{c} x_<\right) \sin\left[\frac{\omega}{c}(l - x_>)\right]}{\left(\frac{\tau\omega}{c}\right) \sin\left(\frac{\omega l}{c}\right)}}$$

This is equivalent to infinite sum expression.

For example $\dfrac{1}{\sin(\omega l / c)}$ has simple poles at $\omega = \pm \omega_n = n\pi \dfrac{c}{l}$