11/10/60

# Lecture 33 Estimates of Lowest Eigenvalue

$$\omega^2[\rho] = \frac{\int_a^b \left[\tau \left(\frac{d\rho}{dx}\right)^2 + V\rho^2\right] dx}{\int_a^b [\sigma \rho^2]\, dx}$$

Let $\rho = \sum_{n=1}^{\infty} a_n \rho_n(x)$

Put in and integrate by parts

$$\omega^2 = \frac{\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} a_n a_q \int \rho_q \left[-\frac{d}{dx}\tau\frac{d}{dx}\rho_n + V\rho_n\right] dx}{\sum_n \sum_q a_n a_q \int \rho_q \sigma \rho_n \, dx}$$

$$-\frac{d}{dx}\tau\frac{d}{dx}\rho_n + V\rho_n = \omega_n^2 \sigma \rho_n$$

$$\int \rho_q \sigma \rho_n \, dx = \delta_{qn}$$

$$\omega^2[\rho] = \sum_n \omega_n^2 a_n^2 \Big/ \sum_n a_n^2$$

Variational estimate suppose $\rho$ is close to $\rho_n$

Chose $\rho = \rho_1 + \sum_{n=2}^{\infty} \epsilon_n \rho_n$

where $\epsilon_n \ll 1$

$$\omega^2 = \left[\omega_1^2 + \sum_{n=2}^{\infty} \epsilon_n^2 \omega_n^2\right] \Big/ \left[1 + \sum_{n=2}^{\infty} \epsilon_n^2\right]$$

$$\omega^2[\rho] = \omega_1^2 + \sum_{n=2}^{\infty} (\omega_n^2 - \omega_1^2)\, \epsilon_n^2$$

(1) $\omega_n^2 - \omega_1^2 > 0$ so $\omega^2[\rho] \geq \omega_1^2$
variational lower bound

(2) If $\epsilon_n$ is small than error in $\omega^2[\rho]$ is only order $\epsilon^2$. Fair estimate for eigenfunction gives very good estimate for eigenvalue

## Green's Functions in 1 dim.

Example String driven by some external force per unit length

$$\mathrm{Re}\ \sigma(x)\, f(x)\, e^{-i\omega t}$$

$$\sigma(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \tau \frac{\partial u}{\partial x} - V u + \mathrm{Re}\ \sigma f e^{-i\omega t}$$

$$U(x,t) = \mathrm{Re}\ u(x)\, e^{-i\omega t}$$

$U(x)$ satisfies inhomogeneous sturm-Liouville eq

$$-\frac{d}{dx} \tau \frac{du}{dx} + V u - \omega^2 \sigma u = \sigma f$$

$\omega$ is frequency of external force and is not in general one of the eigenvalues.

Define $L_0 = -\frac{d}{dx}\left[\tau \frac{d}{dx}\right] + V(x)$

dif. operator

Solve for Green's function in 2 ways

(1) Eigenfunction Expansion

$$L_0 \rho_n = \omega_n^2 \sigma \rho_n$$

Expand $u$ in eigenfunctions

$$u(x) = \sum_{n=1}^{\infty} c_n \rho_n(x) \qquad [L_0 - \omega^2\sigma] u = \sigma f$$

$$\sum_n c_n (\omega_n^2 - \omega^2) \sigma \rho_n = \sigma f$$

Project out coef. $c_m$

$$c_m = \frac{1}{\omega_m^2 - \omega^2} \int_a^b \rho_m(y') \sigma(y') f(y') \, dy'$$

$$u(x) = \sum_{n=1}^{\infty} \int_a^b \left[ \frac{\rho_n(x) \rho_n(y) \sigma(y)}{\omega_n^2 - \omega^2} \right] f(y) \, dy$$

$$= \int_a^b G_\omega(x,y) \sigma(y) f(y) \, dy$$

$$\boxed{G_\omega(x,y) = \sum_{n=1}^{\infty} \left[ \frac{\rho_n(x) \rho_n(y)}{\omega_n^2 - \omega^2} \right]}$$

Integrate over external force with Green's function to get response of system

Let $f(x)\sigma(x) = \delta(x - y_0)$

$$[L_0 - \omega^2 \sigma] u(x) = \delta(x - y_0)$$

$$u(x) = \int_a^b G_\omega(x,y) \delta(y - y_0) \, dy$$

$U(x) = G_\omega(x, y_0)$

$$\boxed{[L_0 - \omega^2 \sigma]\, G_\omega(x, y_0) = \delta(x - y_0)}$$

Properties of $G_\omega(x,y)$

1. It is symmetric $G_\omega(x,y) = G_\omega(y,x)$
2. It has poles at each of the eigenfrequencies $\omega^2 = \omega_n^2$
3. It satisfies b.c. at $x, y = a$ or $x, y = b$ because $\rho_n(x)$ satisfies b.c.

## Second solution for Greens function

If $x \neq y$

$$[L_0 - \omega^2 \sigma]\, G_\omega(x,y) = 0 \qquad a \le x < y$$

Let $G_\omega^<(x,y)$ satisfy b.c. at $x = a$

$$G_\omega^<(x,y) = A\, U_1(x) \qquad a \le x < y$$

because $\omega^2 \neq \omega_i^2$ $U_1(x)$ does not satisfy b.c. at $x = b$.

$$[L_0 - \omega^2 \sigma]\, U_1(x) = 0$$

Note $A$ may depend on $y$

$$[L_0 - \omega^2 \sigma]\, G_\omega^>(x,y) = 0 \qquad y < x \le b$$

$$G_\omega^>(x,y) = B\, U_2(x)$$

Match the two parts at $x=y$

$$-\frac{d}{dx}\tau\frac{dG}{dx}(x,y)+VG(x,y)-\omega^2\sigma G(x,y)=\delta(x-y)$$

Integrate $\int_{y-\epsilon}^{y+\epsilon} dx$

$$-\int_{y-\epsilon}^{\epsilon+y} dx\,\frac{d}{dx}\tau\frac{d}{dx}G+\int_{y-\epsilon}^{y+\epsilon}[V-\omega^2\sigma]G\,dy$$

$$=\int_{y-\epsilon}^{y+\epsilon}dx\,\delta(x-y)$$

We will see below that $G$ is cont. at $x=y$ so $2^{nd}$ integral $\sim$ order $\epsilon$

$$-\tau(y)\frac{d}{dx}G_\omega(x,y)\Big|_{y-\epsilon}^{y+\epsilon}=1$$

$$\left[\frac{dG_\omega^>}{dx}(x,y)-\frac{dG_\omega^<}{dx}(x,y)\right]_{x=y}=-\frac{1}{\tau(y)}$$

$$G^>=B\,U_2(x)$$

$$B\,U_2'(y)-A\,U_1'(y)=-\frac{1}{\tau(y)}$$

also $B\,U_2(y)=A\,U_1(y)$

let $A=-\dfrac{U_2(y)}{\tau(y)\,W[U_1(y),U_2(y)]}$

$$B=-\frac{U_1(y)}{\tau(y)\,W[U_1(y),U_2(y)]}$$

$$W[U_1,U_2]=U_1(x)\,U_2'(x)-U_2(x)\,U_1'(x)$$

Note $\tau(y)\,W[U_1(y),U_2(y)]=\text{const.}=C$

$$[L_0 - \omega^2 \sigma] u_1 = [L_0 - \omega^2 \sigma] u_2 = 0$$

multiply first by $u_2$ and second by $u_1$ and subtract

$$-u_2(x) \frac{d}{dx} \tau \frac{du_1}{dx} + u_1 \frac{d}{dx} \tau \frac{d}{dx} u_2 = 0$$

or

$$\frac{d}{dx} \left\{ \tau \left[ u_1(x) \frac{d}{dx} u_2 - u_2(x) \frac{d}{dx} u_1 \right] \right\} = 0$$

$$\Rightarrow \quad \tau (u_1 u_2' - u_2 u_1') = C$$

$$\boxed{G_\omega(x,y) = -\frac{u_1(x_<)\, u_2(x_>)}{C}}$$

$x_<$ = smaller of $x, y$

$u_1$ satisfies b.c. at $x = a$
$u_2$ satisfies b.c. at $x = b$

$\Rightarrow$ $G_\omega$ satisfies both b.c.
$G_\omega$ is symmetric

$\frac{d}{dx} G_\omega$ is discont. at $x = y$ to give $\delta$ function