11/6/00

Lecture 31 Strings and Sturm-Liouville Theory

Start reading chapter 7

Use simple physics of strings to illustrate some general math methods

Lagrangian for a string

$$L = \int_0^l \mathcal{L}\, dx$$

$$\mathcal{L} = \frac{1}{2}\sigma(x)\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial u}{\partial x}\right)^2$$

$\sigma(x)$ = mass density

$\tau(x)$ = tension

Euler-Lagrange's eq.

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\sigma(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right]$$

If $\sigma(x) = \sigma$ $\tau(x) = \tau$ a constant

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad c = \sqrt{\tau/\sigma}$$

Seek normal mode solutions

$$u(x,t) = C\,\rho(x)\cos(\omega t + \phi$$

$$\frac{\partial^2 \rho}{\partial x^2} + k^2\rho(x) = 0\ , \quad k \equiv \omega/c$$

B.C. $\rho(0) = \rho(l) = 0$

$$\rho_n(x) = \left(\frac{2}{l\sigma}\right)^{1/2} \sin \frac{n\pi x}{l}$$

$$k_n = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots \infty$$

$$\int_0^l dx \; \rho_m(x) \; \sigma \; \rho_n(x) = \delta_{nm}$$

General solution

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n(x) \underbrace{C_n \cos(\omega_n t + \phi_n)}$$

or $$\left[a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi c t}{l}\right)\right]$$

Inital conditions

$$u(x,0) = f(x)$$

$$\dot{u}(x,0) = g(x)$$

$$a_n = \left(\frac{2}{l\sigma}\right)^{1/2} \int_0^l \sin\left(\frac{n\pi x}{l}\right) f(x) \, \sigma \, dx$$

From
$$\int_0^l \rho_n(x) \; \sigma \; u(x,0) \, dx$$

using $$\int_0^l \rho_n \, \sigma \, \rho_m \, dx = \delta_{nm}$$

$$\left(\frac{n\pi c}{l}\right) b_n = \left(\frac{2}{l\sigma}\right)^{1/2} \int_0^l \sin\left(\frac{n\pi x}{l}\right) g(x) \, \sigma \, dx$$

Note normal coordinates

$$\xi_n = c_n \cos(\omega_n t + \phi_n)$$

$$a_n = c_n \cos\phi_n \qquad b_n = -c_n \sin\phi_n$$

$$L = \frac{1}{2} \sum_n \left(\dot{\xi}_n^2 - \omega_n^2 \xi_n^2\right)$$

$$H = \frac{1}{2} \sum_n \left(\dot{\xi}_n^2 + \omega_n^2 \xi_n^2\right)$$

$$H = T + V = \frac{1}{2}\sigma \int_0^l dx \left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\tau \int_0^l dx \left(\frac{\partial u}{\partial x}\right)^2$$

gives same answer

$$H = E = \frac{1}{2} \sum_n \omega_n^2 c_n^2$$

General String(like) Problem

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

$$\mathcal{T} = \frac{1}{2}\sigma(x)\left(\frac{\partial u}{\partial t}\right)^2$$

$$\mathcal{V} = \frac{1}{2}\tau(x)\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}v(x)\,u^2$$

Example of extra $v(x)$ would be massive string under gravity

$$\sigma(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}\right] - v(x)\,u(x)$$

Seek normal modes $u(x,t) = \rho(x)\cos(\omega t + \phi)$

Problem Set 9 Due 11/13/06

1. F+W 7.3

2. F+W 7.8

3. F+W 7.9

$$\boxed{-\frac{d}{dx}\left[\tau(x)\frac{d\rho}{dx}\right] + V(x)\rho = \omega^2 \sigma(x)\rho}$$

known as Sturm - Liouville equation

$\tau(x)$, $V(x)$, $\sigma(x)$ are <u>real</u> in $a \le x \le b$

$\left.\begin{array}{l}\tau(x) > 0 \\ \sigma(x) > 0\end{array}\right\}$ in open interval $a < x < b$

τ and σ could vanish at end points

<u>B.C.</u>

① Fixed ends $\rho = 0$

② Natural boundary condition

$$\tau \frac{d\rho}{dx} = 0$$

③ General homogeneous b.c.

$$\alpha \frac{d\rho}{dx} - \beta\rho = 0$$

④ Periodic b.c.

$$\rho(a) = \rho(b)$$
$$\rho'(a) = \rho'(b)$$

<u>Variational principle</u>

$$\omega^2[\rho] = \frac{\frac{1}{2}\int_a^b \left[\tau(x)\left(\frac{d\rho}{dx}\right)^2 + V\rho^2\right]dx}{\frac{1}{2}\int_a^b [\sigma(x)\rho^2]\,dx} = \frac{I_1}{I_2}$$

Sturm - Liouville eq. is Euler Lagrange eq. for above variational functional

$$\delta \omega^2 = \frac{1}{I_2}\left(\delta I_1 - \omega^2 \delta I_2\right) = 0$$

$$\delta \omega^2 = \frac{\delta I_1}{I_2} - \frac{I_1}{I_2^2}\delta I_2$$

$$\delta I_1 - \omega^2 \delta I_2 = \int_a^b \left[\tau(x)\frac{d\rho}{dx}\frac{d}{dx}\delta\rho + V\rho\,\delta\rho - \omega^2\sigma\rho\,\delta\rho\right]dx = 0$$

Integrate by parts

$$= \delta\rho\,\tau\frac{d\rho}{dx}\Big|_a^b + \int_a^b \left[V\rho - \omega^2\sigma\rho - \frac{d}{dx}\left[\tau\frac{d\rho}{dx}\right]\right]\delta\rho\,dx$$

$\Rightarrow$

$$\boxed{-\frac{d}{dx}\left[\tau\frac{d\rho}{dx}\right] + V(x)\,\rho(x) = \omega^2\sigma(x)\,\rho(x)}$$

Sturm - Liouville eq.

If $\left[\delta\rho\,\tau\,\rho'\right]\Big|_a^b = 0$

True for ① fixed ends $\delta\rho = 0$ ② Natural b.c.

$\tau \rho' = 0$ with arbitrary $\delta\rho$.

3. Periodic b.c.

4. Need small change in functional for mixed b.c. see text.