11/3/06

Lecture 30 Poisson Brackets and QM

The Poisson bracket of two quantities F, G is defined

$$[F, G]_{PB} = \sum_\sigma \left[\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma}\right]$$

$$[F, G]_{PB} = -[G, F]_{PB}$$

$$[H, F]_{PB} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\frac{\partial F}{\partial p_\sigma} - \frac{\partial H}{\partial p_\sigma}\frac{\partial F}{\partial q_\sigma}\right)$$

$$= \sum_\sigma \left(-\dot{p}_\sigma \frac{\partial F}{\partial p_\sigma} - \dot{q}_\sigma \frac{\partial F}{\partial q_\sigma}\right)$$

$$= -\left(\frac{dF}{dt} - \frac{\partial F}{\partial t}\right)$$

Given $F = F(q_\sigma, p_\sigma; t)$

so

$$\boxed{\frac{dF}{dt} = -[H, F]_{PB} + \frac{\partial F}{\partial t}}$$

$$\dot{q}_\sigma = -[H, q_\sigma]_{PB} = \partial H/\partial p_\sigma$$

$$\dot{p}_\sigma = -[H, p_\sigma]_{PB} = -\frac{\partial H}{\partial q_\sigma}$$

Important PB

$$[p_\alpha, q_\beta]_{PB} = \sum_\sigma \left(\frac{\partial p_\alpha}{\partial q_\sigma}\frac{\partial q_\beta}{\partial p_\sigma} - \frac{\partial p_\alpha}{\partial p_\sigma}\frac{\partial q_\beta}{\partial q_\sigma}\right)$$

$$= -\sum_\sigma \delta_{\alpha\sigma}\delta_{\beta\sigma} = -\delta_{\alpha\beta}$$

$$[p_\alpha, p_\beta] = [q_\alpha, q_\beta] = 0$$

Canonical Transformations

Consider transformation $q_\sigma\, p_\sigma \longrightarrow Q_\sigma, P_\sigma$

Invert
$$q_\sigma = q_\sigma(Q_1 \cdots Q_n, P_1 \cdots P_n; t)$$
$$p_\sigma = p_\sigma(Q_1 \cdots Q_n, P_1 \cdots P_n; t)$$

$\sigma = 1, \ldots, n$

Transformation is canonical [leaves form of H equations unchanged] if

$$[P_\alpha, Q_\beta]_{PB} = \delta_{\alpha\beta}$$
$$[P_\alpha\ P_\beta]_{PB} = 0 = [Q_\alpha, Q_\beta]_{PB}$$

Transition to QM

Canonical quantization procedure

Start with $L = L(q_1 \cdots q_n, \dot{q}_1, \cdots \dot{q}_n, t)$

Define $p_\sigma = \dfrac{\partial L}{\partial \dot{q}_\sigma}$ $\quad \sigma = 1, \ldots, n$

$$H(p_1 \cdots p_n, q_1 \cdots q_n, t) \equiv \sum_\sigma p_\sigma \dot{q}_\sigma - L$$

To quantize system define the commutator of two quantities

$$[A, B] \equiv AB - BA$$

Canonical quantization $\quad [A, B]_{PB} \longrightarrow \dfrac{1}{i\hbar}[A, B]$

Note works for any set of generalized coordinates. Require

$$[p_\alpha, q_\alpha] = \frac{\hbar}{i}\delta_{\alpha\beta} \qquad [p_\alpha, p_\beta] = [q_\alpha, q_\beta] = 0$$

again $p_\alpha = \dfrac{\partial L}{\partial \dot{q}_\alpha}$

Commutators takes us far beyond classical mechanics. Need to interpret p, q as operators acting on a Hilbert space of state vectors

In QM

$$\frac{d\hat{F}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{F}] + \frac{\partial \hat{F}}{\partial t}$$

Commutator takes into acount time dep. of state vector from S. equation

$$\hat{H}|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle$$

<u>Example</u> Quantize small amp. osc.

Start with a general Lagrangian

Find place of static equilibrium

$$\left.\frac{\partial V}{\partial q_\sigma}\right|_{q_0} = 0$$

Expand $\qquad q_\sigma = q_\sigma^0 + \eta_\sigma$

$$L = \frac{1}{2} \sum_{\sigma\lambda} (m_{\sigma\lambda} \dot{\eta}_\sigma \dot{\eta}_\lambda - V_{\sigma\lambda} \eta_\sigma \eta_\lambda)$$

Go to normal coordinates

$$\underline{\eta} = \underline{A}\, \underline{\xi}$$

or

$$\underline{\xi} = \underline{A}^{+} \underline{m}\, \underline{\eta}$$

with A modal matrix $A_{\lambda\sigma} = \rho_\lambda^{(\sigma)}$

eigenvector with eigenvalue ω_σ^2

$$L = \frac{1}{2} \sum_{\sigma=1}^{n} (\dot{\xi}_\sigma^2 - \omega_\sigma^2 \xi_\sigma^2)$$

Now quantize

$$p_\sigma = \frac{\partial L}{\partial \dot{\xi}_\sigma} = \dot{\xi}_\sigma$$

Want $[\hat{p}_\sigma, \hat{\xi}_\beta] = \frac{\hbar}{i} \delta_{\sigma\beta}$

Can chose $\hat{p}_\sigma = \frac{\hbar}{i} \frac{\partial}{\partial \xi_\sigma}$

and $\hat{\xi}_\sigma = \xi_\sigma$

or $\hat{p}_\sigma = p_\sigma \quad \hat{\xi}_\sigma = -\frac{\hbar}{i} \frac{\partial}{\partial p_\sigma}$

both sets have

$$[\hat{p}_\sigma, \hat{q}_\beta] = \frac{\hbar}{i} \delta_{\sigma\beta}$$

$$[\hat{p}_\sigma, \hat{p}_\beta] = [\hat{q}_\sigma, \hat{q}_\beta] = 0$$

$$H = \frac{1}{2} \sum_\sigma (p_\sigma^2 + \omega_\sigma^2 \xi_\sigma^2)$$

This is just n uncoupled simple Harmonic osc.

We know energy levels of SHO

$$E = \sum_{\sigma=1}^{n} \hbar\omega_\sigma \left(n_\sigma + \frac{1}{2}\right)$$

Can have different # of quanta n_σ in each of the $\sigma = 1, \ldots, n$ modes

$n_\sigma = 0, 1, \ldots, \infty$

Very general result. Any system undergoing small osc. looks like n uncoupled harmonic osc and can have any # of quanta in each mode.

Properties of system $\rightarrow$ determine n normal mode frequencies ω_σ