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Lecture 29 Hamiltonian Dynamics

Lagrangian is a function of q_σ, $\dot{q}_\sigma$ (and t)

$$L(q, \dot{q}; t)$$

Define generalized momenta

$$p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma} \qquad \sigma = 1, \dots, n$$

assume this is invertable:

$$\dot{q}_\sigma = \dot{q}_\sigma(p_1 \dots p_n; q_1, \dots q_n; t)$$

$$H \equiv \sum_\sigma p_\sigma \dot{q}_\sigma - L(q, \dot{q}; t)$$

$$dH = \sum_\sigma \left(p_\sigma d\dot{q}_\sigma + \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \left(\frac{\partial L}{\partial t}\right) dt$$

$$= \sum_\sigma \left(\dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma \right) - \left(\frac{\partial L}{\partial t}\right) dt$$

Thus
$$H = H(p_1 \dots p_n, q_1 \dots q_n; t)$$

and
$$\left.\frac{\partial H}{\partial t}\right|_{p,q} = -\left(\frac{\partial L}{\partial t}\right)_{q\dot{q}}$$

Legendre transformation from $q, \dot{q} \Rightarrow p, q$

$$dH = \sum_\sigma \left(\left(\frac{\partial H}{\partial p_\sigma}\right) dp_\sigma + \left(\frac{\partial H}{\partial q_\sigma}\right) dq_\sigma \right) + \frac{\partial H}{\partial t}\, dt$$

$$\frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma , \qquad \frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}$$
$$= -\dot{p}_\sigma$$

Hamilton's equations

$$\boxed{\begin{aligned} \dot{q}_\sigma &= \frac{\partial H}{\partial p_\sigma} \\ \dot{p}_\sigma &= -\frac{\partial H}{\partial q_\sigma} \end{aligned}}$$

Set of $2n$ coupled 1^{st} order dif. equations.

Compare to n coupled 2^{nd} order dif equations for Lagrange's equations.

Let $dp_\sigma = \dot{p}_\sigma\, dt \qquad dq_\sigma = \dot{q}_\sigma\, dt$

$$\frac{dH}{dt} = \sum_\sigma \left[\frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma + \frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma \right] + \frac{\partial H}{\partial t}$$

$$= \sum_\sigma \left(\dot{q}_\sigma \dot{p}_\sigma + - \dot{p}_\sigma \dot{q}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$ IF H has no explicit time dep. than $dH/dt = 0$

Hamilton's Equations from Modified Hamilton's Principle

$$\delta \int L\, dt = \delta \int_{t_1}^{t_2} \left[\sum_\sigma p_\sigma \dot{q}_\sigma - H \right] dt$$

Treat p_σ and q_σ on an equal footing. Carry out variations of p_σ and q_σ independently.

$$p_\sigma \to p_\sigma + \delta p_\sigma$$

$$q_\sigma \to q_\sigma + \delta q_\sigma$$

$$\delta p_\sigma(t_1) = \delta p_\sigma(t_2) = 0$$

$$\int_{t_1}^{t_2} dt \left[\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + p_\sigma \delta \dot{q}_\sigma - \frac{\partial H}{\partial p_\sigma} \delta p_\sigma - \frac{\partial H}{\partial q_\sigma} \delta q_\sigma \right) \right] = 0$$

Integrate by parts $\quad p_\sigma \delta \dot{q}_\sigma \to -\dot{p}_\sigma \delta q_\sigma$

$$\int_{t_1}^{t_2} dt \left\{ \sum_\sigma \left[\left(\dot{q}_\sigma - \frac{\partial H}{\partial p_\sigma} \right) \delta p_\sigma - \left(\dot{p}_\sigma + \frac{\partial H}{\partial q_\sigma} \right) \delta q_\sigma \right] \right\} = 0$$

$\Rightarrow$

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \checkmark$$

Canonical Transformations

Lagrangian dynamics allows a wide choice of generalized coordinates

Can transform from one set q_σ to another Q_σ and form of Lagrange's equations is unchanged

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = 0$$

Transformation $Q_\sigma = Q_\sigma(q_1 \cdots q_n; t)$ yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_\sigma} - \frac{\partial L}{\partial Q_\sigma} = 0$$

Same form. In Lagrangian dynamics transformation $q_\sigma \rightarrow Q_\sigma$ is all at a given time.

In Hamiltonian dynamics form of Hamilton's equations is unchanged under a much wide class of transformations.

Hamiltonian dynamics treats q, p on same footing

$$q_\sigma, p_\sigma \longrightarrow Q_\sigma, P_\sigma$$

$$\frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma \qquad \frac{\partial H}{\partial q_\sigma} = -\dot{p}_\sigma$$

$$\Rightarrow \quad \frac{\partial H}{\partial P_\sigma} = \dot{Q}_\sigma, \quad \frac{\partial H}{\partial Q_\sigma} = -\dot{P}_\sigma$$

Transformation is canonical if form of H eq unchanged.

Can chose the new coordinates such that $\frac{\partial H}{\partial P_6}$ and $\frac{\partial H}{\partial Q_6}$ are trivial. Than the transformation from $P_6, Q_6 \leftrightarrow p_6, q_6$ solves the problem.

Hamilton - Jacobi theory solves for the dynamics by finding such a transformation.

Formally very elegant. Howere hard to find example of problem where solution does not look continued.

Hamilton - Jacobi theory has some applications in semi classical quantization and early developement of QM.

Example of Hamiltonian dynamics

Particle in E+M field

$$L = \frac{1}{2} m \dot{r}^2 - e\Phi(\vec{r}, t) + \frac{e}{c} \dot{\vec{r}} \cdot \vec{A}(\vec{r}, t)$$

$$B = \nabla \wedge A \qquad E = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Important to express H in correct variables

$$H = \frac{1}{2} m \dot{\vec{r}}^2 + e\Phi$$

See text. What happened to $\vec{B}$ and $\vec{A}$?

$H(p, q)$ not $H(\dot{q}, q)$

Need $\vec{r} = \vec{r}(p, q)$ $\quad \vec{r} = (x_1, x_2, x_3)$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c} A_i$$

p_i = canonical momentum

$$m\dot{x}_i = p_i - \frac{e}{c} A_i$$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\Phi$$

H written in correct coordinates

See text $\quad \frac{\partial H}{\partial p_i} = \dot{x}_i \quad$ and $\quad \frac{\partial H}{\partial x_i} = -\dot{p}_i$

gives $\quad m\dot{x}_i = p_i - \frac{e}{c} A_i$

$$\dot{p}_i = -e\frac{\partial}{\partial x_i}\Phi + \frac{e}{mc}\sum_j \left(p_j - \frac{e}{c}A_j\right)\frac{\partial}{\partial x_i} A_j$$

$$m\ddot{x}_i = \dot{p}_i - \frac{e}{c}\frac{d}{dt} A_i$$

Leads to Lorentz force law.