Lecture 40  Sound Waves in Fluids

Pressure = Force / Area

Consider pressure forces on a small volume of fluid.

\[ p(x) \rightarrow \begin{array}{c}
\Delta x \\
y \\
x + \Delta x
\end{array} \rightarrow p(x+\Delta x) \]

\[ F_x = p(x) \text{ area} \, dy \, dz - p(x+\Delta x) \text{ area} \, dy \, dz \]
\[ = - \frac{\partial}{\partial x} p \, dx \, dy \, dz = - \frac{\partial p}{\partial x} \, dV \]

In 3 dim: Volume of fluid

\[ \vec{F} = - \vec{\nabla} p \, dV \]

Newton's 2nd law

\[ M = \rho \, dV \]
\[ \rho \, dV \, \frac{\, \vec{dV}}{dt} = - \nabla \rho \, dV \]

or

\[ \frac{\, \vec{dV}}{dt} = - \frac{1}{\rho} \nabla \rho \]

This gives the acceleration of fluid from unbalanced pressure forces.
Total time derivative
\[
\frac{\partial \mathbf{V}}{\partial t} = -\mathbf{V} \cdot \nabla \mathbf{V} + \frac{\partial}{\partial t} \sum_{i} \mathbf{x}_{i} \cdot \frac{\partial \mathbf{V}}{\partial t}
\]

\[
\frac{\partial \mathbf{V}}{\partial t} = -\nabla \mathbf{V} + \nabla \cdot \left( \mathbf{V} \cdot \nabla \mathbf{V} \right)
\]

Much of the complication in Hydrodynamics comes from nonlinear terms. Elements of fluid which start out close together can end up far apart.

Nonlinear term:
\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{1} \nabla p
\]

Continuity equation - Conservation of mass
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0
\]

\[
\mathbf{J} = \text{mass current} = \rho \mathbf{V}
\]

See text. The only way the density can change at a point \( x \) is if the material originally present flowed somewhere else or if new material flowed to \( x \).

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0
\]

Sound waves are small amplitude pressure waves.
They follow directly from eqs (A) and (B). A small change in the pressure leads to a change in density

$$\vec{\nabla} p = \left( \frac{\partial p}{\partial p} \right) \vec{\nabla} p$$

Consider a gas originally at a uniform temperature. A pressure wave can do $\text{p} \cdot \text{d} \text{v}$ work on the gas as it compresses it. This raises the gas's temperature. Therefore, sound waves are not isothermal. However, sound waves are reasonably quick and there is not time for significant heat conduction. Therefore, sound waves are 

adiabatic $\equiv$ constant entropy

The square of the sound speed (gas

we will see it is

$$c^2 = \left( \frac{\partial p}{\partial \rho} \right)_{T}$$

Let $p = p_0 + p'$

where the equilibrium $p_0$ is assumed independent of space and time.

Work to First Order in $p'$ and $\vec{\nabla}$

$$\frac{\text{d} \vec{v}}{\text{d} t} = \frac{\text{d} \vec{v}}{\text{d} t} + (\vec{\nabla} \cdot \vec{V}) \vec{V} = -\frac{1}{\rho_0} c^2 \vec{\nabla} p'$$

$$\frac{\text{d} \vec{v}}{\text{d} t} \approx -\frac{1}{\rho_0} c^2 \vec{\nabla} p'$$

\( \star \)
\[ \frac{\partial \rho}{\partial t} = \frac{\partial \rho'}{\partial t} \]

\[ \frac{\partial \rho'}{\partial t} + \nabla \cdot \rho_0 \nabla \rho' = 0 \]

Differentiate this equation w.r.t. time

\[ \frac{\partial^2 \rho'}{\partial t^2} + \nabla \cdot \rho_0 \frac{\partial \nabla \rho'}{\partial t} = 0 \]

Use eq. (C) for \( \frac{\partial \nabla \rho'}{\partial t} \)

\[ \frac{\partial^2 \rho'}{\partial t^2} = \nabla \cdot \rho_0 \left( \frac{1}{\rho_0} \nabla \rho' \right) \]

\[ \frac{\partial^2 \rho'}{\partial t^2} = \nabla \cdot c^2 \nabla \rho' \]

If \( c^2 = \left( \frac{\partial p}{\partial \rho} \right)_s \) is independent of position

\[ \frac{1}{c^2} \frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 \rho' \]

Then dim. wave eq. which shows

\[ c = \left[ \left( \frac{\partial p}{\partial \rho} \right)_s \right]^{1/2} \]

Is the sound velocity.
Boundary conditions

1. Fixed walls

Fluid can't move against walls so

\[ \mathbf{n} \cdot \mathbf{V} = 0 \]

at surface. Here \( \mathbf{n} \) is normal to surface

Write \( \mathbf{V} = -\frac{1}{c^2} \frac{\partial \Phi}{\partial t} \) for sound waves

with \( \nabla \times \mathbf{V} = 0 \)

This defines velocity pot. \( \Phi \). Easy to show

\[ \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 0 \]

same form of wave eq.

Also \( \rho^* = \frac{\partial \Phi}{\partial t} \)

\[ \frac{\partial \Phi}{\partial t} = -\frac{1}{P_0} \nabla P \]

\[ \Rightarrow \quad P = P_0 \frac{\partial \Phi}{\partial t} + \text{const.} \]

Write \( P = P_0 + P' \) from sound wave

\[ P' = P_0 \frac{\partial \Phi}{\partial t} \]

Equilibrium pressure
Think of a small cavity open to a much larger volume. Any small amount of "air" leaking into the larger volume will not increase the pressure of the large volume significantly ⇒

\[ \frac{\partial P}{\partial t} = 0 \] 

at the opening.

Free surface.