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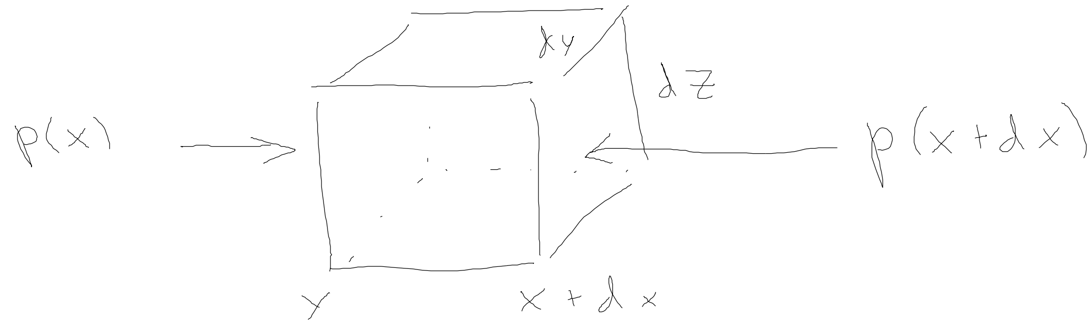
# Lecture 40 Sound Waves in Fluids

Pressure $\equiv$ Force / Area

Consider pressure forces on a small volume of fluid




$$F_x = p(x)\,\underbrace{dy\,dz}_{\text{area}} - p(x+dx)\,dy\,dz$$

$$\approx -\frac{\partial p}{\partial x}\,dx\,dy\,dz = -\frac{\partial p}{\partial x}\,dV$$

Volume of fluid

In 3 dim

$$\vec{F} = -\vec{\nabla} p \; dV$$

Newton's 2nd law $\quad m = \rho\, dV$

$$\rho\, dV \frac{d\vec{V}}{dt} = -\vec{\nabla} p \; dV$$

or

$$\boxed{\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla} p}$$

This gives the acceleration of fluid from unbalanced pressure forces.

Note total time derivative

$$\frac{d}{dt}V_j = \frac{\partial V_j}{\partial t} + \sum_i \frac{\partial V_j}{\partial x_i}\frac{dx_i}{dt}$$

or

$$\boxed{\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}}$$

Much of complication in Hydrodynamics comes from nonlinear 2nd term. Elements of fluid which start out close together can end up far apart.

$\boxed{N2}$

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{1}{\rho}\nabla p \qquad (A)$$

Continuity equation - Conservation of mass

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$\vec{J}$ = mass current = $\rho\vec{V}$

See text. The only way the density can change at a point $\vec{x}$ is if the material originaly present flowed somewhere else or if new material flowed to $\vec{x}$.

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{V} = 0} \qquad (B)$$

<u>Sound waves</u>

are small amplitude pressure waves.

They follow directly from eqs (A) + (B).
A small change in the pressure leads to a change in density

$$\vec{\nabla} p = \left(\frac{\partial p}{\partial \rho}\right) \vec{\nabla} \rho$$

Consider a gas originally at a uniform temperature. A pressure wave can do $p\,dV$ work on the gas as it compresses it. This raises the gases temperature. Therefore sound waves are not isothermal. However sound waves are reasonably quick and there is not time for significant heat condution. Therefore sound waves are adiabatic $\equiv$ constant entropy

The square of the sound speed (as we will see) is

$$\boxed{c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S}$$

Let $\rho = \rho_0 + \rho'$

where the equillibrium $\rho_0$ is assumed independent of space + time.

Work to first order in $\rho'$ and $\vec{V}$

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V} = -\frac{1}{\rho_0} c^2 \vec{\nabla} \rho'$$

$$\boxed{\frac{\partial \vec{V}}{\partial t} \simeq -\frac{1}{\rho_0} c^2 \vec{\nabla} \rho'} \qquad (*)$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho'}{\partial t}$$

$$\frac{\partial \rho'}{\partial t} + \vec{\nabla} \cdot \rho_0 \vec{V} = 0$$

differentiate this equation w.r.t. time

$$\frac{\partial^2 \rho'}{\partial t^2} + \vec{\nabla} \cdot \rho_0 \frac{\partial \vec{V}}{\partial t} = 0$$

Use eq. $\circledast$ for $\frac{\partial \vec{V}}{\partial t}$

$$\frac{\partial^2 \rho'}{\partial t^2} = \vec{\nabla} \cdot \rho_0 \frac{1}{\rho_0} c^2 \vec{\nabla} \rho'$$

$$\frac{\partial^2 \rho'}{\partial t^2} = \vec{\nabla} \cdot c^2 \vec{\nabla} \rho'$$

If $c^2 \equiv (\partial p/\partial \rho)_s$ is independent of position

$$\boxed{\frac{1}{c^2} \frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 \rho'}$$

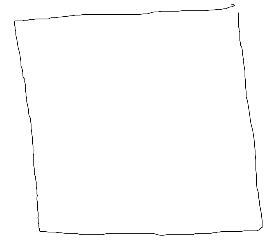
Three dim. wave eq. which shows

$$c = \left[ \left( \frac{\partial p}{\partial \rho} \right)_s \right]^{1/2}$$

is the sound velocity.

Boundary conditions

(1) Fixed walls



Fluid can't move against walls so

$$\hat{n} \cdot \vec{v} = 0$$

at surface. Here $\hat{n}$ is normal to surface

Write $\vec{v} = -\vec{\nabla}\Phi$ for sound waves with

$$\vec{\nabla} \wedge \vec{v} = 0$$

This defines velocity pot. $\vec{V}$. Easy to show

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \nabla^2 \Phi$$

For fixed walls

$$\boxed{\hat{n} \cdot \vec{\nabla}\,\Phi = 0}$$

Same form of wave eq.

Also ~~p =~~ $\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p$

or $-\vec{\nabla}\frac{\partial}{\partial t}\Phi = -\frac{1}{\rho_0}\nabla p$

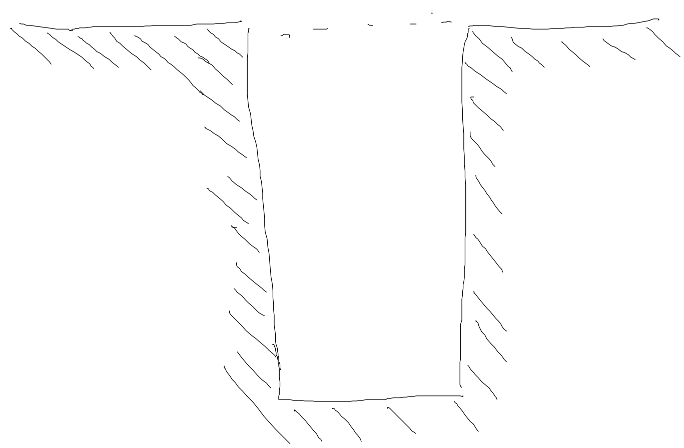
$$\Rightarrow \quad p = \rho_0 \frac{\partial \Phi}{\partial t} + \text{const.}$$

Write $p = p_0 + p'$ ← from sound wave

$p_0$ ← Equilibrium pressure

$$\boxed{p' = \rho_0 \frac{\partial \Phi}{\partial t}}$$

② Free surface



Think of a small cavity open to a much large volume. Any small amount of "air" leaking into large volume will not increase the pressure of the large volume significantly $\Longrightarrow$

$p' \approx 0$ at the opening

$\boxed{\dfrac{\partial \Phi}{\partial t} \approx 0}$ free surface.