Lecture 37 Sound Waves in Fluids

Strings and membranes introduce basic concepts of continuum mechanics with simple low dimensionality systems.

Now start hydrodynamics. Much harder and jaunt points on a membrane stay adjacent but fluid elements can separate.

**Pressure**

\[ p = \text{force/area} \]

Force is normal to an area and independent of orientation.

Newton's 2nd law \( F_x \), fluid element

\[
F_x = p(x) A - p(x+dx) A
\]

\[
F_x = -\frac{\partial p}{\partial x} \, dV
\]

\( dV = A \, dx \)
Force per unit volume or fluid
\[ \vec{F}_{pr} = -\vec{\nabla} p \]
Pressure force

Let there also be an applied force per unit volume
\[ \vec{F}_{app} = \int \vec{F}_{app} \, dV \]
\[ \vec{F}_{app} = \text{Force per unit mass} \]
\[ p = \text{density} = \text{mass of fluid per volume} \]

In equilibrium
\[ (p \vec{F}_{app} - \vec{\nabla} p) \, dV = 0 \]
\[ \vec{F}_{app} = \frac{\vec{\nabla} p}{p} \]
Fundamental equation of hydrostatics

Example: gravity
\[ \vec{F}_{app} = -g \hat{z} \]
\[ \frac{\partial p}{\partial z} = -g \rho \]

Newton's 2nd law
\[ \int p \, dV \frac{d\vec{V}}{dt} = -\nabla p \, dV + p \vec{F}_{app} \, dV \]

Total deviation of velocity
\[ \Delta \vec{v}_i(x, t) = \sum_j \frac{\partial \Delta \vec{v}_i}{\partial x_j} + \frac{\partial \Delta \vec{v}_i}{\partial t} \]
\[ \frac{dv_i}{dt} = \sum_j \frac{dx_j}{dt} \frac{dv_i}{dx_j} + \frac{\mathbf{v}_i}{\mathbf{t}} \]

Rate of change of velocity of a given element of a fluid.

\[ \frac{dv}{dt} = \nabla (\frac{1}{2}v^2) - \nabla (\nabla \cdot \mathbf{v}) \]

Use vector identity:

\[ (\nabla \cdot \mathbf{v}) \mathbf{v} = \nabla (\frac{1}{2}v^2) - \nabla (\nabla \cdot \mathbf{v}) \]

See book

\[ \frac{dv}{dt} = \frac{\mathbf{j}}{\rho} - \frac{1}{\rho} \nabla p \]

Newton's second law

Note: Fundamental differential equations of hydrodynamics are nonlinear.

Leads to interesting physical phenomena and great mathematical difficulties.

Exact solutions very rare.

Continuity equation:

Mass current density, \( \mathbf{j} = \rho \mathbf{v} \)
Conservation of mass (see text)
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{p} + \rho \nabla \cdot \mathbf{v} = 0 \]

For a uniform incompressible fluid both \( \frac{\partial \rho}{\partial t} = 0 \) and \( \nabla \cdot \mathbf{p} = 0 \), i.e., \( \nabla \cdot \mathbf{v} = 0 \). For incompressible flow, divergence of \( \mathbf{v} \) is zero so material will not be able to build up or leave a test volume and density will change.

Conservation of Momentum

Stress Tensor and Euler's equation

Euclidian description

Focus on a given volume element \( V \) in \( V = \text{ fluid} \)
\[ \Delta \mathbf{v} \cdot \frac{d}{dt} \int V \mathbf{v} \cdot dV = \Delta t \int V \frac{\partial}{\partial t} (\rho \mathbf{v}) \cdot dV \]

For a nonviscous fluid, momentum can change in only 3 ways:

1) A volume force \( \mathbf{F} \) (drop label \( \text{app} \) for)
2) The pressure over a force
\[ -\mathbf{p} d\mathbf{A} \] on each surface element enclosing \( V \)
3) Concentrate on a fixed volume

\[ \Delta t \int \Delta \vec{A} \cdot \nabla p \, \vec{V}_i = -\Delta t \int \Delta A \cdot \nabla_j v_i \cdot \vec{V}_i + \nabla \times \vec{P}_i \]

\[ \frac{d}{dt} \int V \vec{P}_i = -\int \sum_j \int A \bar{T}_{ij} + \int \nabla \times \vec{P}_i \]

Decom stress tensor

\[ T_{ij} = \rho \delta_{ij} + \rho \vec{V}_i \cdot \vec{V}_j \]

\[ \frac{d}{dt} \int V \rho \vec{V}_i = -\int \sum_j \int A \bar{T}_{ij} + \int \nabla \times \vec{P}_i \]

Use divergence theorem to rewrite surface

\[ \frac{\partial}{\partial t} \left( \rho \vec{V}_i \right) + \sum_j \frac{\partial}{\partial x_j} T_{ij} = \rho \vec{f}_i \]

Above three equations plus continuity

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \]

gives 4 equations for the five unknowns

\[ \rho, \rho, \vec{V} \]

Subject with equation of state \( p(\rho, T) \) or \( p(\rho, s) \)
If \( T \) (isothermal) or \( S \) is constant then have 5 equations in 5 unknowns.

\[
\frac{\partial}{\partial t} \int_{V} \left( \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} + p \mathbf{v} \right) dV = \int_{V} \frac{\partial}{\partial t} \left( \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} + p \mathbf{v} \right) dV
\]

Use \( \frac{\partial}{\partial t} \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{f} \)

and \( \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \)

\[
\frac{\partial}{\partial t} \left( \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} \right) = \frac{\partial}{\partial t} \left( \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} \right) + \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{v} = -\frac{1}{\rho} \mathbf{v} \cdot \mathbf{v} \nabla p + \mathbf{f} \cdot \mathbf{v}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} \right) = -\nabla \cdot \left[ \frac{\rho}{2} \mathbf{v} \cdot \mathbf{v} \right] - \nabla p \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{f}
\]

Now for internal energy

\[
M \frac{dt}{dt} - p dV = \frac{\rho}{\rho_0} M \mathbf{v} \cdot \mathbf{v} + \frac{\rho}{\rho_0} M \mathbf{v} \cdot \mathbf{v}
\]

Increase in internal energy as seen by a nonmoving observer is work done on fluid. Only work from change in volume

Assume isentropic process. Nonviscous flow is typically non-dissipative either because there is no mechanism for heat generation or because motion is too rapid for heat conduction.