Lecture 21  Torque Free Motion

Last time Euler's equations in body fixed frame:

\[ I_1 \frac{d\omega_1}{dt} = \omega_2 \omega_3 (I_2 - I_3) + \tau_{1e} \]
\[ I_2 \frac{d\omega_2}{dt} = \omega_3 \omega_1 (I_3 - I_1) + \tau_{2e} \]
\[ I_3 \frac{d\omega_3}{dt} = \omega_1 \omega_2 (I_1 - I_2) + \tau_{3e} \]

Read sections in text on Compound pendulum and billiards. (Note, billiards lab. is not required)

Torque Free motion.

In general Euler's equations are complicated because you have to figure out where body axes point.

\[ \tau_{ei} \] not so bad, but \[ \hat{\tau}_{ei} \] is complicated because \[ \hat{e}(t) \] is needed.

Simple case \[ \tau_{ei} \equiv 0 \] so \[ \hat{\tau}_{ei} \equiv 0 \]

...
Torque Free Symmetric Top

Symmetric top \( I_1 = I_2 \neq I_3 \)

Example: Earth is flattened by rotation.

\[ I_1 = I_2, \quad I_3 > I_1 \]

\[ \frac{(I_3 - I_1)}{I_1} \sim \frac{1}{365} \]

\[ I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_3 - I_1) \]

\[ I_1 \dot{\omega}_2 = \omega_3 \omega_1 (I_2 - I_1) \]

\[ I_3 \dot{\omega}_3 = 0 \]

\( \omega_3(t) = \omega_3 \) constant

Define \( \tau = \omega_3 (I_3 - I_1) / I_1 \)

\[ \dot{\omega}_1 = -\tau \dot{\omega}_2 \]

\[ \dot{\omega}_2 = \tau \dot{\omega}_1 \]

\[ \dot{\omega}_1 = -\omega_2 \dot{\omega}_2 = -\omega^2 \omega_1 \]

\( \omega_1(t) = \omega_1^0 \cos \tau t \)

\( \omega_2(t) = \omega_1^0 \sin \tau t = -\frac{1}{\tau} \dot{\omega}_1 \)
Define $\lambda$ so
\[
\omega_1^0 = \omega \sin \lambda
\]
\[
\omega_3^0 = \omega_3 = \omega \cos \lambda
\]

$\omega$ precesses about the $\beta$ axis with a period
\[
\Omega = \frac{2\pi}{\frac{1}{I_1} - \frac{1}{I_3}} \text{ 1 day}
\]

$\sim 365$ days

Angle $\lambda \sim 6 \times 10^{-7}$ rad.

Note Earth is not a perfect rigid motion body.

Torque-Free Motion: Asymmetric Top

$I_1 \neq I_2 \neq I_3$

Example book

$I_3 > I_2 > I_1$
General solution is complicated. However, can use two constants of motion

\[ E = T = \frac{1}{2} \left( I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right) \]

and angular momentum components of \( \mathbf{L} \) are complicated because axes change. Instead

\[ L = I_1 \omega_1 + I_2 \omega_2 + I_3 \omega_3 \]

is simple. Use \( E \) equation and \( L \) equation to express \( \omega_1 \) and \( \omega_2 \) as functions of \( \omega_3 \).
Then Euler's equations yield a complicated definite integral \( \Phi \). Solution involves elliptic integrals.

Let us instead consider small amplitude motion like chapter 4 about one axis

\[ \omega_1 = \eta_1 \eta_1 \quad \omega_2 = \eta_2 \eta_2 \quad \omega_3 = \omega_0 + \eta_3 \eta_3 \]

with the \( \eta_i \) small.

\[ I_1 \eta_1 = (I_2 - I_3) \omega_0 \eta_2 \quad I_2 \eta_2 = (I_3 - I_1) \omega_0 \eta_1 \]

\[ I_3 \eta_3 = \eta_1 \eta_2 (I_1 - I_2) = 0 + O(\eta^2)\]
\[ I_1 \dot{\gamma}_1 = (I_2 - I_3) \omega_0 \qquad \dot{\gamma}_2 = \frac{(I_3 - I_2)}{I_2} \omega_0 \qquad \dot{\gamma}_1 = \frac{(I_2 - I_3)}{I_1} \frac{(I_3 - I_1)}{I_2} \omega_0^2 \qquad \dot{\gamma}_2 = -\Omega_c \gamma_1 \]

\[ \Omega_c^2 = \left( \frac{I_3 - I_1}{I_1} \right) \left( \frac{I_3 - I_2}{I_2} \right) \omega_0^2 \]

- **Stable** if \( \Omega_c^2 > 0 \) 
  - Smallest moment of inertia

- **Unstable** if \( I_3 > I_1 > I_2 \) 
  - Middle moment of inertia

Example: \( I_3 = 2I_1 < 2I_2 \)

Start rotating around 3 to transfer all angular momentum to axis 2. Have to rotate twice as fast.

\[ T = \frac{I_3 \omega_3^2}{2} + \frac{I_2 \omega_2^2}{2} \]

Rotating twice as fast with \( I_2 \) only \( \frac{1}{2}I_1 \) will double kinetic energy. Don't have enough kinetic energy to transfer 5
Likewise if you start rotating about
$I_3$ to transfer kinetic energy
you need to rotate at $\sqrt{2}$
the original speed. However then
$L = I_3 w_3 = (2 I_2) \frac{w_2}{\sqrt{2}} = \sqrt{2} I_2 w_2$

You don't have enough angular momentum
from small to big moment of
inertia.

Solution Transfer $L$ to the middle
of inertia and $I$ to be smaller

Euler Angles

Need way to specify orientation
Warning other books use different
conventions.

1) Rotate about $e_3$
2) Rotate about line of
cnodes by angle $\beta$
3) Rotate about new $e_2$
   by angle $\gamma$
Generalized coordinates are Euler angles $\alpha$, $\beta$, $\gamma$.

Notes

(a) Line of nodes is $L$ to plane of $\mathbf{e}_2$, $\mathbf{e}_3$.

(b) To specify orientation of axis $\mathbf{e}_2$, need two coordinates $\alpha$, $\beta$.

$\beta$ is polar angle $\phi$.

$\alpha$ is $\theta$ of spherical coordinates.

(c) Angle $\gamma$ specifies orientation of body around $\mathbf{e}_3$. 