Lecture #20 Rigid bodies continued

Way back in section 2 we separate motion about center of mass

\[ T = T_{cm} + T' \]

\[ T_{cm} = \frac{1}{2} M V_{cm}^2 \]

\[ T' = \frac{1}{2} \sum_i m_i V_i^2 \quad V_i = \text{Velocity w.r.t. center of mass} \]

Above is true for any system. If we have a rigid body then

\[ T_{cm} = \frac{1}{2} \sum_{i<j} I_{ij} \omega_i \omega_j \]

Where \( I_{ij} \) is inertia tensor about \( cm \) and \( \omega_i, \omega_j \) are rotational velocities about axes through center of mass.

Problem 5.1

If \( Q \) is fixed have two choices

1) Rotation about \( Q \) where \( I_{ij} \) is calculated with parallel axis theorem
2) Full center of mass motion + rotation about \( cm \).
If point \( G \) moves then one can't calculate motion of \( G \) with

\[
T = \frac{1}{2} M V^2 + \frac{1}{2} I \omega^2
\]  

(\text{Equation A})

Instead one must calculate \( V_{cm} \) and rotation about \( cm \)

\[
T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega_{cm}^2
\]

Equation A fails because there are cross terms.

\[
\vec{V}_p = \vec{V}_A + \vec{V}_p
\]

velocity of point \( p \) relative to \( A \)

\[
T = \frac{1}{2} \sum m_p \vec{V}_p^2 = \frac{1}{2} \sum m_p \left( \vec{V}_A + \vec{V}_p \right)^2
\]

\[
= \frac{1}{2} \left( \sum m_p \right) \vec{V}_A^2 + \frac{1}{2} \sum m_p \vec{V}_p^2 + \frac{1}{2} \sum m_p \vec{V}_p \cdot \vec{V}_p
\]

This last term does not vanish in general.

However if \( \vec{V}_p \) is with respect to \( cm \) then

\[
\sum m_p \vec{x}' = 0
\]

definition of \( cm \)

so

\[
\sum m_p \frac{d\vec{x}'_p}{dt} = 0 = \sum m_p \vec{V}_p
\]
Principal Axes

Diagonalize \( I_{ij} \)

\[
\det \left| I_{ij} - \lambda \delta_{ij} \right| = 0 \quad \text{for } i, j = 1, 2, 3
\]

As eigenvalues \( \lambda \) are principal moments of inertia

\( e_i \) are eigenfunctions \( i = 1, 2, 3 \)

Normalization of unit vectors \( e_i \)\( e \): 1

Modal matrix \( A = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_1 & e_2 & e_3 \\ e_1 & e_2 & e_3 \end{bmatrix} \)

\[
A^T A = I = A^T A
\]

\[
A^T I A = \lambda D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
\]

Diagonal.

In original arbitrary body axis the instantaneous angular velocity is

\[
\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}
\]
Define a new vector \( y \)

\[
\mathbf{w} = \mathbf{A} \mathbf{s} \quad \mathbf{s} = \mathbf{A}^T \mathbf{w}
\]

Components of \( \mathbf{w} \) along the principal axes

\[
T = \frac{1}{2} \mathbf{w}^T \mathbf{I} \mathbf{w} = \frac{1}{2} \mathbf{y}^T \mathbf{A}^T \mathbf{I} \mathbf{A} \mathbf{y} = \frac{1}{2} \sum_{i=1}^{2} \mathbf{y}_i^2 = \frac{1}{2} \sum_{i=1}^{2} \mathbf{w}_i^2 = T
\]

\[
\mathbf{l}_{new} = \mathbf{A}^T \mathbf{L} = \begin{bmatrix}
    -2
    0
    -1
\end{bmatrix}
\]

\[
\mathbf{l}_{new} = \mathbf{A}^T \mathbf{I} \mathbf{w} = \mathbf{A}^T \mathbf{I} \mathbf{A} \mathbf{s} = \lambda \mathbf{s}
\]

\[
\mathbf{l}_{new} = \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix}
\]

\[
\mathbf{I}_5 = (\mathbf{A}^T \mathbf{I} \mathbf{A})_{55} = 5 \rho^2 \frac{\rho(r)}{r^3} \sqrt{r^2 - \left( \frac{r}{\rho} \mathbf{e}_s \right)^2}
\]
Euler's Equations

\[ \left( \frac{d\mathbf{T}}{dt} \right)_{\text{initial}} = \nabla \times \mathbf{L} = \mathbf{p}^{(e)} \]  

and equation holds in two cases

1) If the origin is fixed in some initial frame then holds with \( \mathbf{L} \) and \( \mathbf{p}^{(e)} \) computed about that point.

2) Equation (B) is also true if \( \mathbf{L} \) and \( \mathbf{p}^{(e)} \) are computed about center of mass. In this case all of the internal torques cancel.

If \( \mathbf{\dot{L}} \) write

\[ \left( \frac{d\mathbf{\dot{L}}}{dt} \right)_{\text{initial}} = \left( \frac{d\mathbf{\dot{L}}}{dt} \right)_{\text{body}} + \mathbf{W} \times \mathbf{\dot{L}} = \mathbf{p}^{(e)} \]

From chapter two general relation between time derivative of any vector.
\[ \frac{d\mathbf{I}}{dt}_{\text{body}} = \mathbf{\tau}^{(e)} - \hat{\omega} \mathbf{I} \]

Chose principal body axes

\[ \mathbf{\tau}^{(e)} = \mathbf{\tau}^{(e)} \otimes \mathbf{e} \]
\[ \mathbf{L}_5 = \mathbf{I} \otimes \mathbf{e}, \quad \mathbf{\omega} \otimes \mathbf{w} \]
\[ \mathbf{L}_5 = \mathbf{I}_s \mathbf{w}_5 \quad \text{along principal axes} \]

\[ I_1 \frac{d\omega_1}{dt} = \mathbf{\tau}^{(e)}_1 - (\hat{\omega}_1 \mathbf{L}_1) \]
\[ I_2 \frac{d\omega_2}{dt} = \mathbf{\tau}^{(e)}_2 + \omega_3 \omega_1 (I_3 - I_1) \]
\[ I_3 \frac{d\omega_3}{dt} = \mathbf{\tau}^{(e)}_3 + \omega_1 \omega_2 (I_1 - I_2) \]

Euler's equations

They are very elegant and compact. To compact, they involve a funny set of dynamical variables \( \omega_i \) and driving torques \( \mathbf{\tau}^{(e)}_i \), which are w.r.t. the principal axes of a body which is tumbling in space.
So we will charge just because the body is changing its orientation.

To calculate the $\mathbf{P}(\theta)$, one needs to know the orientation of the body and to find the orientation of the body you need the solution of Euler's equations for its motion.

Euler's equations are most useful if

1. $\dot{\vec{n}} = 0$

or if the principal axes orientation is partially constrained.