Lec #17  Continuous Systems

Define normal coordinate

\[ \phi_n(t) = C_n \cos(\omega_n t + \phi_n) \quad n=1, 2, \ldots, \infty \]

\[ u(x, t) = \sum_n \phi_n(x) \phi_n(t) \]

Put into Lagrangian

\[ V = \frac{\mu}{2} \int_0^L \left( \frac{d^2 u}{dx^2} \right)^2 dx = \frac{\mu}{2} \int_0^L \frac{d^2 u}{dx^2} \frac{du}{dx} \frac{du}{dx} \]

\[ \frac{d^2 \phi_n}{dx^2} = -k_n \phi_n \quad k_n = \frac{n \pi}{L} \]

\[ V = -\frac{\mu}{2} \int_0^L dx \sum_n \phi_n(x) \phi_n(x) \sum_n k_n \phi_n(x) \phi_n(x) \]

\[ \int_0^L \phi_n \phi_n' = \frac{1}{2} \delta_{nn'} \]

\[ V = \frac{\mu}{2} \sum_n k_n \phi_n^2(t) \quad \omega_n = \frac{n \pi}{L} \sqrt{\frac{\mu}{k_n}} \]

\[ V = \frac{1}{2} \sum_n \omega_n^2 \phi_n^2(t) \]

\[ L = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \phi_n^2 - \omega_n^2 \phi_n^2(t) \right] \]

Lagrangian for a cantilever string can be written as a discrete sum over an infinite number of modes.

Lagrangian's eq. \[ \ddot{\phi}_n = -\omega_n^2 \phi_n \]
Hamilton's Principle for Continuous System

\[ S = \int L \, dt = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial \dot{x}} \dot{x} - L \right) \, dt \]

\[ L = \text{Lagrangian density} \quad [\text{Energy/Length}] \]

\[ L = \frac{1}{2} \left[ \sigma (\dot{u})^2 - 2 \left( \frac{\partial u}{\partial x} \right)^2 \right] \quad \text{for string} \]

\[ u \rightarrow u(x,t) + S u(x,t) \]

\[ S(x,t_1) = S(x,t_2) = 0 \quad \text{all } x \]

\[ S(0,t) = S(L,t) \quad \text{all } t \]

\[ 0 = S \int L \, dt = S \int \left( \frac{\partial L}{\partial \dot{x}} \dot{x} \right) dt + S \int L \, dt \]

Integrate by parts and use B.C.

\[ = S \int dt \left( \frac{\partial L}{\partial x} \right) \, dx - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{x}} \right) \left. \right|_{x=0}^{x=L} + \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) \left. \right|_{t=t_1}^{t=t_2} \]

\[ \delta u = \text{arbitrary } \delta u \]

\[ \frac{\partial \delta u}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial \delta u}{\partial \dot{x}} \right) \left. \right|_{x=0}^{x=L} - \frac{\partial}{\partial t} \left( \frac{\partial \delta u}{\partial \dot{x}} \right) \left. \right|_{t=t_1}^{t=t_2} = 0 \]

Note partial derivation:

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial \dot{x}} \right) \text{ at fixed } u, \frac{\partial u}{\partial t} \]

Then:

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial \dot{x}} \right) \text{ at fixed } t \]
\[ L = \frac{1}{2} m \left( \frac{d\mathbf{u}}{dt} \right)^2 - \frac{1}{2} (2\pi)^2 \left( \frac{\partial \mathbf{u}}{\partial x} \right)^2 \]

\[ \frac{\partial E}{\partial \mathbf{u}} = 0 \quad \frac{\partial E}{\partial \mathbf{u}} = -6 \frac{\partial \mathbf{u}}{\partial t} - \frac{\partial^2 \mathbf{u}}{\partial x^2} \]

\[ -6 \frac{\partial^2 \mathbf{u}}{\partial x^2} + 2 \frac{\partial^2 \mathbf{u}}{\partial x^2} = 0 \]

\[ 0 = \frac{1}{c^2} \frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\partial^2 \mathbf{u}}{\partial x^2} \quad \text{Wave Eq.} \]

\[ c^2 = \frac{\lambda}{\mu} \]