Spherical Waves Lec 9 Multipole Expansion 2/10/11

start with scalar wave eq.

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x,\omega)\, e^{-i\omega t}\, d\omega$$

$$[\nabla^2 + k^2]\,\psi(x,\omega) = 0 \qquad k = \omega/c$$

Expand in spherical coordinates

$$\psi(x,\omega) = \sum_{l\,m} f_{lm}(r)\, Y_{lm}(\theta\phi)$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{l(l+1)}{r^2}\right] f_l(r) = 0$$

let $f_l(r) = \frac{1}{r^{1/2}}\, u_l(r)$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right] u_l(r) = 0$$

Bessel equation 3.75 of order $\nu = l+\frac{1}{2}$

$$f_{lm}(r) = \frac{A_{lm}}{r^{1/2}}\, J_{l+1/2}(kr) + \frac{B_{lm}}{r^{1/2}}\, N_{l+1/2}(kr)$$

Define spherical Bessel functions j_l, n_l
and spherical Hankel functions $h_l^{1,2}$

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{l+1/2}(x)$$

$$n_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} N_{l+1/2}(x)$$

$$h_l^{1,2}(x) = \left(\frac{\pi}{2x}\right)^{1/2}\left[J_{l+1/2}(x) \pm i\, N_{l+1/2}(x)\right]$$

for real x, $h_\ell^2 = h_\ell^{1\,*}$

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{\partial}{\partial x}\right)^\ell \frac{\sin x}{x}$$

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{\partial}{\partial x}\right)^\ell \frac{\cos x}{x}$$

$$j_0 = \frac{\sin x}{x} \qquad j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_2 = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - 3\frac{\cos x}{x^2}$$

$$n_0 = -\frac{\cos x}{x} \qquad n_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_\ell(x) \to \frac{x^\ell}{(2\ell+1)!!} \qquad x \ll \ell$$

$$(2\ell+1)!! = (2\ell+1)(2\ell-1)(2\ell-3)\cdots$$

$$n_\ell(x) \to -\frac{(2\ell-1)!!}{x^{\ell+1}}$$

$$j_\ell(x) \to \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right) \qquad x \gg \ell$$

~~The Wronskian~~
General Solution of $[\nabla^2 + k^2]\,\psi(x,\omega) = 0$ is

$$\psi(x) = \sum_{\ell m}\left[A^1_{\ell m}\, h^{(1)}_\ell(kr) + A^2_{\ell m}\, h^{(2)}_\ell(kr)\right] Y_{\ell m}(\theta,\phi) \quad (*)$$

Look at Green func.

$$[\nabla^2 + k^2]\, G(x, x') = -\delta(x - x')$$

$$G = \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}$$

Spherical wave expansion

$$G(x,x') = \sum_{lm} g_l(r,r')\, Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{l(l+1)}{r^2}\right] g_l = -\frac{1}{r^2}\delta(r-r')$$

Solution that is regular at origin and satisfies outgoing wave b.c. for large r is

$$g_l(r,r') = A\, j_l(kr_<)\, h^{(1)}_l(kr_>)$$

Correct discont. in slope at $r=r'$ if $A = ik$

$$\frac{e^{ik|x-x'|}}{4\pi|\vec{x}-\vec{x}'|} = ik \sum_{l=0}^{\infty} j_l(kr_<)\, h^{(1)}_l(kr_>) \sum_{m=-l}^{l} Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)$$

take $k \to 0$ as check

$$j_l(kr_<) \sim \frac{(kr_<)^l}{(2l+1)!!} \qquad h^1_l(x) = j_l(x) + i\, n_l(x)$$

$$h^{(1)}_l(kr_>) \sim i\, n_l(kr_>) \Longrightarrow -i\frac{(2l-1)!!}{(kr_>)^{l+1}}$$

$$\frac{1}{4\pi|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{(2l-1)!!}{(2l+1)!!} \frac{r_<^l}{r_>^{l+1}} \sum_m Y^*_{lm} Y_{lm}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{lm} \left(\frac{4\pi}{2l+1}\right) \frac{r_<^l}{r_>^{l+1}} \cancel{\sum_m} Y^*_{lm} Y_{lm} \qquad \checkmark\ (3.70)$$

Now look at angular functions in QM

$$\vec{L} = \frac{1}{i}(\vec{r} \times \vec{\nabla})$$

$$L^2 Y_{lm} = l(l+1)\, Y_{lm} = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}$$

$$L_+ = L_x + iL_y = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_- = L_x - iL_y = e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_z = -i\frac{\partial}{\partial\phi}$$

$$\vec{r}\cdot\vec{L} = 0 \qquad \text{since} \quad L = \frac{1}{i}(\vec{r}\times\nabla)$$

$$L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)}\, Y_{l\,m+1}$$

$$L_- Y_{lm} = \sqrt{(l+m)(l-m+1)}\, Y_{l\,m-1}$$

$$L_z Y_{lm} = m Y_{lm}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ ) - \frac{L^2}{r^2}$$

Multipole Expansion of E+M Fields

For Harmonic time dep. Maxwell eqs

$$\nabla\times E = ikZ_0 H \qquad \nabla\times H = -ikE/Z_0$$
$$\nabla\cdot E = \nabla\cdot H = 0$$

$$\Rightarrow \quad (\nabla^2 + k^2)H = 0, \quad \nabla\cdot H = 0, \quad E = \frac{iZ_0}{k}\nabla\times H$$

or $\quad [\nabla^2 + k^2]E = 0, \quad \nabla\cdot E = 0, \quad H = -\frac{i}{kZ_0}\nabla\times E$

Find multipole solutions for E, H. Consider

$$\nabla^2(r\cdot\vec{A}) = r\cdot\nabla^2 A + 2\nabla\cdot A$$

$$(\nabla^2 + k^2)(\vec{r}\cdot\vec{E}) = 0 \qquad (\nabla^2 + k^2)(r\cdot H) = 0$$

define magnetic multipole of order (l, m)

$$r\cdot H^m_{lm} = \frac{l(l+1)}{k} g_l(kr)\, Y_{lm}(\theta\phi)$$

$$r\cdot E^m_{lm} = 0$$

$$g_\ell(kr) = A^1_\ell h^1_\ell(kr) + A^2_\ell h^2_\ell(kr) \quad \text{from } *$$

Note $\ell(\ell+1)/k$ is for convience

$$\cancel{i}kZ_0 \; r\cdot H = \frac{1}{i}\vec{r}\cdot(\nabla\times E) = \frac{1}{i}(r\wedge\nabla)\cdot E = L\cdot E$$

$$L\cdot\vec{E}^m_{\ell m} = \cancel{k}Z_0\,\ell(\ell+1)\,g_\ell(kr)\,Y_{\ell m}(\theta\phi)$$

$$r\cdot E^m_{\ell m} = 0$$

guess $E^m_{\ell m} = Z_0 g_\ell(kr) L\, Y_{\ell m}(\theta\phi)$

than $L\cdot E = Z_0\, g_\ell(kr) L^2 Y_{\ell m} = \ell(\ell+1)Z_0\, g_\ell Y_{\ell m}$

and $r\cdot L = 0$ so $r\cdot E^m_{\ell m} = 0$

So

$$\boxed{\vec{E}^m_{\ell m} = Z_0\, g_\ell(kr)\, L\, Y_{\ell m}(\theta\phi)}$$

$$\boxed{\vec{H}^m_{\ell m} = \frac{-i}{kZ_0}\nabla\wedge E^m_{\ell m}}$$

Fields for magnetic multipoles of order ℓ, m

The fields on an electric multipole of order (ℓ, m)

$$r\cdot E^E_{\ell m} = -Z_0\frac{\ell(\ell+1)}{k} f_\ell\, Y_{\ell m}$$

$$r\cdot H^E_{\ell m} = 0$$

$$\Rightarrow \quad H^E_{\ell m} = f_\ell(kr)\, L\, Y_{\ell m}$$

$$E^E_{\ell m} = \frac{iZ_0}{k}\nabla\wedge H^E_{\ell m} \qquad f_\ell(kr) = A^1_\ell h^1_\ell + A^2_\ell h^2_\ell(kr)$$

Introduce normalized vector spherical harmonics

$$\vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L}\, Y_{lm} \qquad \equiv 0 \text{ for } l=0$$

$$\int \vec{X}^*_{l'm'} \cdot \vec{X}_{lm}\, d\Omega = \delta_{ll'}\, \delta_{mm'}$$

General solution to Maxwell equations

$$H = \sum_{lm} \left[a_E(l,m)\, f_l(kr)\, \vec{X}_{lm} - \frac{i}{k} a_m(l,m)\, \nabla \wedge g_l(kr) \vec{X}_{lm} \right]$$

$$E = Z_0 \sum_{lm} \left[\frac{i}{k} a_E(l,m)\, \nabla \wedge f_l \vec{X}_{lm} + a_m\, g_l(kr)\, \vec{X}_{lm} \right]$$

Consider outgoing wave b.c. appropriate for a localized source.

$$g_l(kr),\ f_l(kr) \rightarrow h_l^{(1)}(kr)$$

Work in $r \rightarrow \infty$ limit

$$h_l^1 \sim (-i)^{l+1} \frac{e^{ikr}}{kr}$$

$$\cancel{\nabla \wedge (-i)^{l+1} \frac{e^{ikr}}{kr} \vec{X}_{lm} = (-i)^{l+1} \nabla \wedge \vec{L} \frac{e^{ikr}}{kr} Y_{lm}}$$

book says $\nabla \times L = \frac{1}{i} \left[\vec{r}\, \nabla^2 - \vec{\nabla} \left(1 + r \frac{\partial}{\partial r} \right) \right]$

$$\cancel{r \frac{\partial}{\partial r} \frac{e^{ikr}}{kr} = ikr \frac{e^{ikr}}{kr} = i\, e^{ikr} + O\left(\frac{1}{r}\right)}$$

$$\cancel{\nabla \times L\, \frac{e^{ikr}}{kr} Y_{lm} = - \vec{\nabla} \left[e^{ikr} Y_{lm} \right]}$$

$r \longrightarrow \infty$

$$H^E_{lm} = \cancel{\alpha_E(l,m)} \frac{-i^{l+1}}{kr} e^{ikr} \frac{\vec{L}}{\sqrt{l(l+1)}} Y_{lm}$$

$$E^E_{lm} = \frac{Z_0}{\sqrt{l(l+1)}} \frac{i}{k} \nabla_\wedge \frac{(-i)^{l+1}}{kr} e^{ikr} \vec{L} Y_{lm}$$

$$= \frac{Z_0}{\sqrt{l(l+1)}} \frac{-i^l}{k^2} \left\{ \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) \wedge \vec{L} Y_{lm} + \frac{e^{ikr}}{r} \vec{\nabla}_\wedge \vec{L} Y_{lm} \right\}$$

$$= \cancel{\frac{Z_0}{\sqrt{l(l+1)}} \frac{-i^{l+1}}{k} \hat{n} \wedge \vec{L} Y_{lm}}$$

$$= \frac{Z_0}{\sqrt{l(l+1)}} \frac{(-i)^l}{k^2} \left\{ \frac{e^{ikr}}{r} (ik) \hat{n} \wedge \vec{L} Y_{lm} + \frac{e^{ikr}}{r} \frac{1}{i} \left[\vec{r} \nabla^2 - \vec{\nabla} \right] Y_{lm} \right\}$$

Y_{lm} only depends on angles so

$$\nabla^2 Y_{lm} \rightarrow \frac{l(l+1)}{r^2} Y_{lm}$$

$$\vec{\nabla} Y_{lm} \rightarrow O \frac{1}{r}$$

So last term in [ ] is $O(1/r)$

$$E^E_{lm} = -Z_0 (-i)^{l+1} \frac{e^{ikr}}{kr} \hat{n} \wedge \frac{L Y_{lm}}{\sqrt{l(l+1)}}$$

$$\boxed{\text{end } 2/10/11}$$

$$E^E_{lm} = Z_0 \vec{H}^E_{lm} \wedge \hat{n}$$