2/8/11

Lec. 8 Magnetic Dipole / Electric Quadrupole Radiation

Last time

$$A(x) = \frac{\mu_0}{4\pi}\int d^3x' \frac{J(x')}{|\vec{x}-\vec{x}'|} e^{ik|\vec{x}-\vec{x}'|}$$

Note retarded green's func. $\rightarrow e^{ik|\vec{x}-\vec{x}'|}$ term

$H = \frac{1}{\mu_0}\nabla\times A$ $\quad E = \frac{iZ_0}{k}\nabla\times H$ $\quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

Assume localized source $x' \leq d$
and $kd \ll 1$

Near field $d \ll r \ll \lambda$, $\quad \lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$

intermediate zone $d \ll r \sim \lambda$

far (radiation) zone $d \ll \lambda \ll r$

In far zone $|\vec{x}-\vec{x}'| \simeq r - \hat{n}\cdot\vec{x}'$

$$A \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' J(x') e^{-ik\hat{n}\cdot\vec{x}'}$$

Electric dipole radiation from $e^{-ik\hat{n}\cdot x'} \simeq 1$

$$\int d^3x' J(x') = -i\omega\int \vec{x}'\rho\, d^3x'$$

from conti. eq. $= -i\omega\vec{p}$

$A = -\frac{i\mu_0}{4\pi}\omega\vec{p}\frac{e^{ikr}}{r}$, $\quad S = \frac{1}{2}E\times H^*$

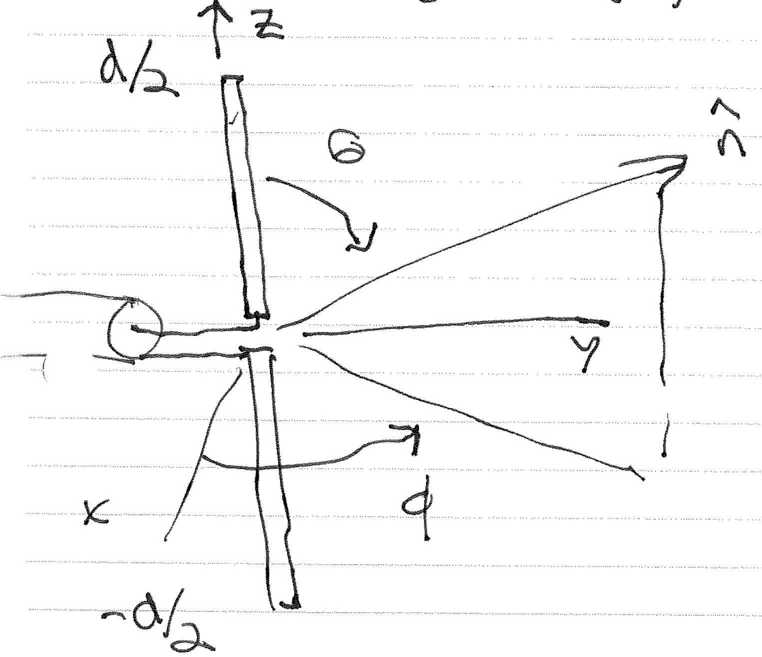
$$\frac{dP}{d\Omega} = \vec{S}\cdot\frac{d\vec{A}}{d\Omega} = \frac{1}{2}\mathrm{Re}(r^2\hat{n}\cdot E\times H^*) = \frac{c^2k^4Z_0}{32\pi^2}|\hat{n}\times\vec{p}\times\hat{n}|^2$$

$$= \frac{c^2k^4Z_0}{32\pi^2}|p|^2\sin^2\theta, \quad P = \int\frac{dP}{d\Omega}\sin\theta\, d\theta\, 2\pi$$

$$P = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2$$

This depends on ω^4 given $k = \frac{\omega}{c}$ for fixed $\vec{p}$. However $\vec{p}$ may depend on ω.

Example center-fed, linear antenna of length $d \ll \lambda$.



$$I(z) e^{-i\omega t} \simeq I_0 \left[1 - \frac{2|z|}{d} \right] e^{-i\omega t}$$

$\cancel{J = I}$

Current ~~is~~ falls to zero at ends of antenna approx. linear

Charge density per unit length ρ'

$$\rho'(z) = \pm \frac{2 i I_0}{\omega d} \qquad \begin{array}{ll} + & \text{for } z > 0 \\ - & \text{for } z < 0 \end{array}$$

from $\nabla \cdot J = i\omega\rho$

$$p = \int_{-d/2}^{d/2} z\rho' \, dz = i \frac{I_0 d}{2\omega}$$

$$\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{128\pi^2} (kd)^2 \sin^2\theta$$

Total power radiated is

$$P = \frac{Z_0 I_0^2 (kd)^2}{48\pi}.$$

For given current I_0, power goes as square of frequency at least in long wavelength domain

$$kd \ll 1$$

$$P \equiv R_{rad} \frac{I_0^2}{2}$$

R_{rad} = radiation resistance of the antenna

$$= \frac{Z_0 (kd)^2}{24\pi} \approx 5 (kd)^2 \text{ Ohms}$$

Magnetic Dipole and Electric Quadrupole fields

We have as $kr \to \infty$

$$A \approx \frac{\mu_0}{4\pi}\left(\frac{e^{ikr}}{r}\right) \int J(x')\, e^{-ik\,\hat{n}\cdot\vec{x}'}\, d^3x'$$

$1 - ik\, n\cdot x' + \ldots$

Next term is

$$A \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik) \int \vec{J}(x')\,(\hat{n}\cdot\vec{x}')\, d^3x'$$

In general this term is smaller by $\sim kd$ compared to 1 term.

We will see that this next term gives rise to magnetic dipole and electric quadrupole radiation. This can be very important when electric dipole moment is zero.

Can rewrite

$$(n \cdot \vec{x}')\, J = \underbrace{\tfrac{1}{2}\left[n \cdot x' J + n \cdot J\, x' \right]}_{\text{rewrite as elec. quad.}} + \underbrace{\tfrac{1}{2}\left[x' \times J \wedge n \right]}_{\text{mag. dipole moment!}}$$

$$m = \int \mathcal{M}\, d^3x = \frac{1}{2}\int x \times J\, d^3x .$$

Keep only this term and

$$A(\vec{x}) = \frac{ik\mu_0}{4\pi}\,(\hat{n} \times \vec{m})\,\frac{e^{ikr}}{r} .$$

$$H = \frac{1}{4\pi}\, k^2\, \frac{e^{ikr}}{r}\,(n \wedge m) \wedge n + O\left(\frac{1}{r^2}\right)$$

$$E = -\frac{Z_0}{4\pi}\, k^2\,(n \wedge m)\,\frac{e^{ikr}}{r}$$

$$\frac{1}{2}\int d^3x' \left[(n \cdot x')\, J + (n \cdot J)\, x' \right]$$

Can be rewritten

$$J_i = \sum_k \left(\frac{\partial}{\partial x'_k} \vec{x}'_i \right) J_k$$

$$= \frac{1}{2}\int d^3x' \left[\sum_l (n \cdot \vec{x}') \frac{\partial}{\partial x'_l} \vec{x}'\, J_l + \sum_l \left(n \cdot \frac{\partial \vec{x}'}{\partial x'_l} \right) J_l\, \vec{x}' \right]$$

Integrate by parts

$$= -\frac{1}{2}\int d^3x' \Big[(\vec{\nabla}\cdot\vec{J})(\vec{n}\cdot\vec{x}')\vec{x}'$$

$$+ \vec{x}'\vec{J}\cdot\nabla'(n\cdot\vec{x}')$$

$$+ (n\cdot\vec{x}')\vec{x}'(\nabla\cdot J) + \underbrace{\vec{x}' J\cdot\vec{\nabla}'(n\cdot\vec{x}')}_{\vec{x}'\,\vec{J}\cdot n} \Big]$$

and use $\nabla\cdot J = i\omega\rho$

$$\frac{1}{2}\int\left[(n\cdot x')J + n\cdot J\,x'\right] d^3x' = -\frac{i\omega}{2}\underbrace{\int \vec{x}'\; n\cdot\vec{x}'\;\rho(x')\,d^3x'}_{\text{related to electric } Q}$$

$$A = -\frac{\mu_0 c k^2}{8\pi}\frac{e^{ikr}}{r}\int \vec{x}'\; n\cdot\vec{x}'\;\rho(x')\,d^3x'$$

$$H = ik\; n\times A/\mu_0$$

$$E = ikZ_0\,(n\times A)\times n/\mu_0$$

Define quadrupole moment tensor

$$Q_{\alpha\beta} = \int\left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)\rho(x)\,d^3x$$

$$H = \frac{-ick^3}{8\pi}\frac{e^{ikr}}{r}\int (n\times x')(n\cdot x')\,\rho(x')\,d^3x'$$

So $n\times\int x'\,n\cdot x'\,\rho\,d^3x' = \frac{1}{3}\,\hat{n}\wedge Q(n)$

vector $Q_\alpha = \sum_\beta Q_{\alpha\beta}\,\hat{n}_\beta$

$$H = \frac{-ick^3}{24\pi} \frac{e^{ikr}}{r} \quad n \wedge Q(n)$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 \left| [n \wedge Q] \wedge n \right|^2$$

$$|n \wedge Q \wedge n|^2 = Q^* \cdot Q - |n \cdot Q|^2$$

$$= \sum_{\alpha\beta\gamma} Q^*_{\alpha\beta} Q_{\alpha\gamma} n_\beta n_\gamma - \sum_{\alpha\beta\gamma\delta} Q^*_{\alpha\beta} Q_{\gamma\delta} n_\alpha n_\beta n_\gamma n_\delta$$

Calculate total power radiated

$\int d\Omega \, |n \wedge Q \wedge n|^2$ use

$$\rightarrow \int n_\beta n_\gamma \, d\Omega = \frac{4\pi}{3} \delta_{\beta\gamma}$$

$$\int n_\alpha n_\beta n_\gamma n_\delta \, d\Omega = \frac{4\pi}{15} \left[\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right]$$

$$\int |n \wedge Q \wedge n|^2 d\Omega = 4\pi \left\{ \frac{1}{3} \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 - \frac{1}{15} \sum_\alpha Q^*_{\alpha\alpha} \sum_\gamma Q_{\gamma\gamma} + 2 \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 \right\}$$

$Q_{\alpha\beta}$ is traceless $\Rightarrow \sum_\gamma Q_{\gamma\gamma} = 0$

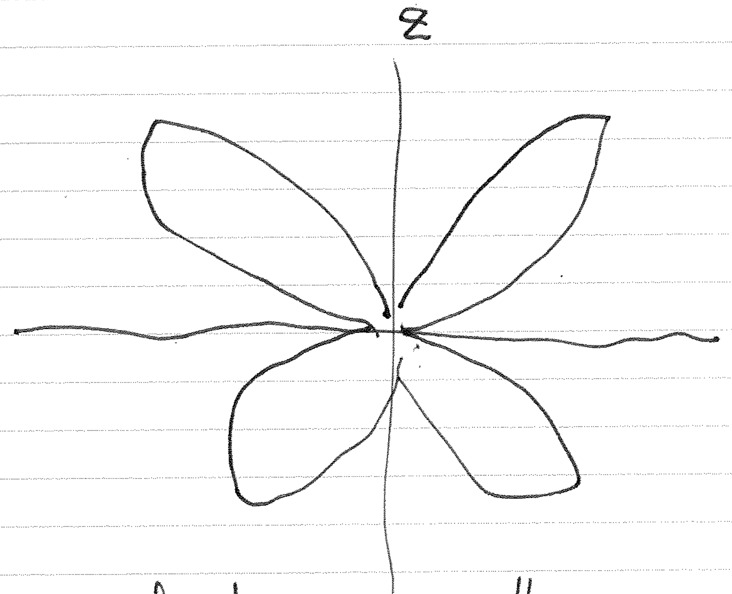
$$P = \frac{c^2 Z_0 k^6}{\pi^2} \frac{4\pi}{1152} \left(\frac{1}{3} - \frac{2}{15} \right) \sum_{\alpha\beta} |Q_{\alpha\beta}|^2$$

$$\boxed{P = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{\alpha\beta} |Q_{\alpha\beta}|^2}$$

In general electric Qud. radiation a factor of $(kd)^2$ smaller than electric dipoles rad.

and

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0^2 k^6}{512\pi^2} Q_0^2 \sin^2\theta \cos^2\theta$$



Qudrupole radiation pattern.

In principle can go to higher orders in expansion of

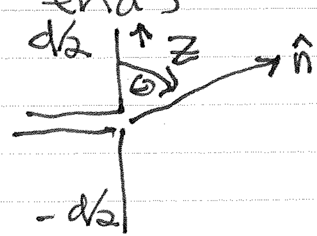
$$e^{-i|\vec{k}|\hat{n}\cdot\vec{x}'}$$

but very painfull. We will develope vector sperical harmonic formalism.

Center-Fed Linear Antenna

Some geometries are simple enough that one can calculate integral over J directly without multipole expansion

Example: linear antenna assume sin distribution of curent that goes to zero at ends



$$\vec{J}(\vec{x}) = I \sin\left[k\left(\frac{d}{2} - |z|\right)\right] \delta(x)\,\delta(y)\,\hat{z}\;\Theta\left(\frac{d}{2} - |z|\right)$$

$$A = \hat{z}\,\frac{\mu_0}{4\pi}\,\frac{I e^{ikr}}{r}\int_{-d/2}^{d/2}\sin\left(\frac{kd}{2}-k|z|\right)e^{-ikz\cos\theta}\,dz$$

$$= \hat{z}\,\frac{\mu_0}{4\pi}\,\frac{2Ie^{ikr}}{kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\frac{kd}{2}}{\sin^2\theta}\right]$$

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\frac{kd}{2}}{\sin\theta}\right]^2$$

Note we did not assume $kd \ll 1$

Example half wavelength antenna $kd = \pi$

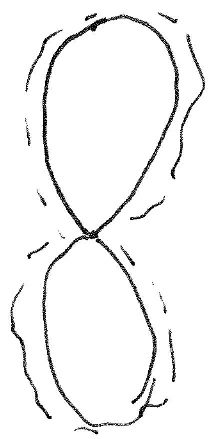
$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\,\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

In dipole approx. we had

$$\frac{dP}{d\Omega} \simeq \frac{Z_0 I_0^2}{128\pi^2}(kd)^2\sin^2\theta = \frac{Z_0 I_0^2}{128}\sin^2\theta$$

Radiation pattern θ, $kd = \pi$
exact solid,
dipole dotted

Figure 9.7



<u>Spherical Wave Solutions of Wave eq.</u>

Systematic multipole expansion of radiation fields