Last time \( n \sin i = n' \sin i' \)

\( i = n' \)

\[ \frac{E''}{E_0} = \frac{n \cos i - \frac{\mu}{\mu'} (n'^2 - n^2 \sin^2 i)^{1/2}}{n \cos i + \frac{\mu}{\mu'} (n'^2 - n^2 \sin^2 i)^{1/2}} \]

For polarization \( \perp \) to plane of \( k, \hat{n} \)

\[ \frac{E''}{E_0} = \frac{\mu}{\mu'} n^2 \cos i - n \sqrt{n'^2 - n^2 \sin^2 i} \]

\[ \frac{E''}{E_0} = \frac{\mu}{\mu'} n^2 \cos i + n (n'^2 - n^2 \sin^2 i)^{1/2} \]

For polarization \( \parallel \) to plane

At optical frequencies \( n, n', \mu, \mu' \) real, \( \mu/\mu' = 1 \)

Polarization by reflection

For polarization \( \parallel \) to plane of incidence there is an angle (Brewster's angle) with no reflected wave.

\[ n^2 \cos i = n (n'^2 - n^2 \sin^2 i)^{1/2} \]
\[n^2 \cos^2 i = n \sqrt{n^4 - n^2 \sin^2 i}\]

\[
\left(\frac{n'}{n}\right)^2 \cos^2 i = \left[\left(\frac{n'}{n}\right)^2 - \sin^2 i\right]^{1/2}
\]

\[x = \frac{n'}{n}\]

\[x^2 \cos^2 i = (x^2 - \sin^2 i)^{1/2}\]

\[x^4 \cos^2 i = x^2 - \sin^2 i = x^2 + \cos^2 i = 1\]

\[x^4 (1-x^4) \cos^2 i = 1-x^2\]

\[(1-x^2)(1+x^2) \cos^2 i = 1-x^2\]

\[1+x^2 \cos^2 i = 1\]

\[\cos^2 i = 1/(1+x^2)\]

or \[\tan i = x = \frac{n'}{n}\]

\[\sin^2 i = \frac{1+x^2}{1+x^2} - 1 = \frac{x^2}{1+x^2}\]

\[\sin i / \cos i = \tan i = x = \frac{n'}{n}\]
At this angle, reflected wave is polarized 100% I to plane of incidence.

Pond

Polarizing sun glasses block horizontal polarization

Total internal reflection

If \( n > n' \) then for some angle

\[ n \sin i = n' \sin r \]

does not have a solution for real \( r \)
\[
\gamma = \frac{\pi}{2}
\]
when
\[
i_0 = \sin^{-1} \frac{n'}{n}
\]

For \( i > i_0 \) there is no refracted wave and all of the power is reflected.

**Example**: Optical Fibre

---

**Dispersion**

Real life materials have \( n = n(\lambda) \) this has many practical implications.

**Example**: Chromatic aberrations

In optics a lens brings different colors to a focus at different places.

\[
\sin i = n(\lambda) \sin r
\]

---

Note: Normal dispersion most materials have \( n \) increase with frequency for visible light.

If first medium is air \( n = \frac{1}{1.0003} \).

Major problem for early telescope designs

1. Minimize length so telescopes became very long and unwieldy.
2. Invention of achromatic telescope
Strongly magnifying lens of weakly dispersive glass

Weak demagnifying lens of strongly dispersive glass

The chromatic aberrations of first lens can cancel those aberrations from second lens and both red and blue can come to focus at same place.

3) Newton invented a reflecting telescope to cure chromatic aberrations

Inset angle of incidence = angle of reflection

Independent of angle. A curved mirror brings all colors to a focus at same point.

Curved mirrors

Flat mirrors

Red and blue focus at same point.
Problems: (a) no good reflecting surfaces in Newton's day, surface of mirror to much high accuracy than lens.

In late 19th / early 20th century reflectors replaced most refractors because mirrors can be supported from back surface while lenses can only be supported around edges. Can make much bigger mirrors.

Largest refractor 40" diameter lens.

7.5 Frequency Dispersion Characteristics of Dielectrics Conductors, Plasmas

Simple model for $E(w)$, assume $\mu \approx \mu_0$

Consider an electron bound by a harmonic force

$$m\ddot{x} + \omega_0^2 x = -eE(x,t)$$

$$\ddot{E}(x,t) = \frac{e}{m} \omega_0 e^{i \omega t}$$

$$\vec{p} = -e\vec{x} = \frac{e^2}{m} \left[ \omega_0^2 - \omega^2 - i \omega \right] \vec{E}$$

Note: ignore screening (low density gas)

$N$ molecules per unit volume with $Z$ electrons per molecule.

Assume collection of frequencies $\omega_i$.

Sum rule $\sum f_i = Z$ $f_i$: osc. strength.
\[ \varepsilon = \varepsilon_0 \left( 1 + \chi_e \right) \]

\[ \rho = \varepsilon_0 \chi_e E \]

\[ \chi_e = \frac{\varepsilon_0 \varepsilon_n^2}{\frac{\varepsilon_0 M}{m}} \left[ \omega^2 - \omega^2 - i \omega \zeta_i \right]^{-1} \]

\[ \varepsilon(\omega) = \frac{\varepsilon_0}{\varepsilon_0 - i \frac{\varepsilon_0 M}{m} \left[ \omega^2 - \omega^2 - i \omega \zeta_i \right]} \]

\[ n = \sqrt{\frac{\varepsilon_M}{\varepsilon_0}} \]

\[ \varepsilon_{\text{eff}}(\omega) = \left[ 1 + \frac{N \varepsilon^2}{\varepsilon_0 \omega} \sum_{i} \frac{f_i}{\omega^2 - \omega^2 - i \omega \zeta_i} \right]^{1/2} \]

Consider only two frequencies

Assume \( \zeta_i \) is small

\[ \text{实} \varepsilon \]

\[ \text{虚} \varepsilon \]
Anomalous dispersion and resonant absorption $v_i$ generally small.

Factor \( \frac{1}{\omega^2 - \omega_i^2} \) is positive for $\omega < \omega_i$ and negative for $\omega > \omega_i$.

At low frequency below lowest $\omega_i$ all terms have same frequency sign and are positive and $\varepsilon(\omega) > 1$. As $\omega$ increases, more and more terms become negative so that $\varepsilon < 1$ near any $\omega_i$. $\varepsilon$ depends rapidly on frequency.

Normal dispersion: $\varepsilon(\omega)$ increases with $\omega$.

Anomalous dispersion: $\varepsilon(\omega)$ decreases with $\omega$ and only happens near $\omega_i$. Im $\varepsilon$ only important near $\omega_i$.

Attenuation of a plane wave in terms of real and imaginary parts of wave vector $k$

\[
k = \sqrt{\mu \varepsilon} \omega = \sqrt{\mu_0 \varepsilon} \omega \]

\[
k = \frac{\omega}{c} n(\omega) \quad \text{Eq. (7.5)}
\]

\[
k = \beta + i \alpha \Rightarrow \text{intensity } \propto \varepsilon
\]

\[
\beta^2 = \frac{\omega^2}{c^2} - \frac{\omega_0^2}{\varepsilon_0} \quad \text{Re} \varepsilon
\]

\[
\beta \propto \frac{\omega_0^2}{c^2} \text{Im} \varepsilon / \varepsilon_0
\]
\[ \alpha = \frac{\text{Im } \varepsilon}{\text{Re } \varepsilon} \]
\[ \beta = \frac{\text{Re } \varepsilon}{\omega} \]

Fractional decrease in intensity per wavelength divided by \( \alpha \) is \( \frac{\text{Im } \varepsilon}{\text{Re } \varepsilon} \)

Note: Dispersion directly related to absorption, \( \text{Im } \varepsilon \)

Low Frequency Behavior, Electrical Conductivity

In limit \( \omega \to 0 \), qualitative difference in response depending on if one of \( \omega J \) is 0 or not.

Assume some fraction \( f_0 \) of electrons act "free" \( \omega = 0 \)

\[ \varepsilon(\omega) = \varepsilon_b(\omega) + i \frac{N e^2 f_0}{m\omega(\varepsilon_0 - i\omega)} \]

This can be related to electrical conductivity \( \sigma \).

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

and assume Ohm's law \( \mathbf{J} = \sigma \mathbf{E} \)

for a normal dielectric \( \varepsilon_b \)

\[ \nabla \times \mathbf{H} = -i\omega \left[ \frac{\varepsilon_b + i\sigma}{\omega} \right] \mathbf{E} \]

Compare this to \( \varepsilon(\omega) \) and deduce
\[ \sigma = \frac{N e^2 f_0}{m (\omega - i \omega)} \]

Model of Drude (1900). \( f_0 \) N is # of free electrons per unit volume in medium.

Bigger the damping \( \sigma \) \( \rightarrow \) smaller the conductivity.

High Frequency limit, Plasma Frequency

At frequencies far above highest \( \omega \)

\[ \frac{\epsilon(\omega)}{\epsilon_0} \rightarrow 1 - \frac{\omega_p^2}{\omega^2} \]

\[ \omega_p^2 = \frac{\sum N e^2 f_i}{\epsilon_0 m} = \frac{N Ze^2}{\epsilon_0 m} \]

Plasma Frequency \( \omega_p \).

The wave # is given in this limit

\[ k = \sqrt{m \epsilon} \omega = \frac{1}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \]

\[ c k = \sqrt{\omega^2 - \omega_p^2} \]

or sometimes \( \omega^2 = \omega_p^2 + c^2 k^2 \) the dispersion relation \( \omega = \omega(k) \)

In dielectric media above true only for \( \omega^2 \gg \omega_p^2 \).
In some situations such as ionosphere or in a tenuous electronic plasma in lab, damping is negligible and above holds true, in the wide range of \( \omega \) even for \( \omega < \omega_p \).

For \( \omega < \omega_p \), \( k \) is purely imaginary ( \( \leq \omega_0 \)).

\[
\alpha_{\text{plasma}} = \frac{2\omega_p}{c}
\]

For a laboratory plasma of density

\[ \sim 10 \text{ to } 10^2 \text{ electrons } / \text{m}^3 \]

\( \omega_p \sim 6 \times 10^{10} \text{ to } 6 \times 10^{12} \text{ s}^{-1} \)

\( \alpha^{-1} \sim 0.2 \text{ to } 2 \times 10^{-3} \text{ cm} \)

For static or low frequency fields.