4/26/11

Lec. 25 Review

Last time photon mass term

$$\mathcal{L}_{Proca} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha$$

mass term

Euler Lag. eq. assuming $\partial^\alpha A_\alpha = 0$

$$\Box A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha$$

Static limit

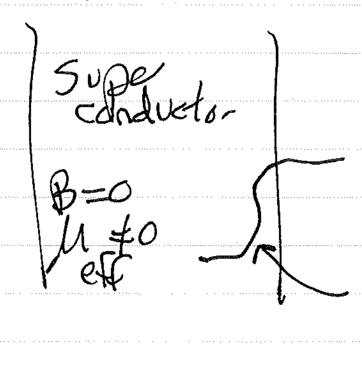
$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha$$

For a charge at rest

$$A_0 = \Phi = q \frac{e^{-\mu r}}{r} \quad \text{falls exp.}$$

In ~~medium~~ superconducting medium photon has an effective mass because of its interactions with charge carriers.

This causes magnetic field to be expeled from (Type I) superconduc. with an exp. screening distance



Normal region

$B \neq 0$, $\mu = 0$

B decays exp. at surface

~~If source is a pt. charge at rest at the origin only time comp. of A_α is nonzero~~

$$A_0 = \Phi \qquad \Phi(\vec{x}) = q\,\frac{e^{-\mu r}}{r}$$

~~Characteristic feature of photon mass is an exp. fall off of range $1/\mu$.~~

~~This alters the character of the earth's magnetic field enough to allow stringent limits on the photon mass from geomagnetic data.~~

Effective "photon" mass in superconduc.
London penetration depth

London theory of electromagnetic behavior of superconductors.

explains Meissner effect (1933)
expulsion of magnetic field from interior of superconductor

Phenomenalogical discussion of current in superconductor.

Assume

$$\vec{J} = Q\, n_Q \vec{v}$$

Q = charge of current carriers

n_Q = density of carriers

In presence of E+M fields rewritten in terms of

$$\vec{p} = m_Q \vec{v} + \frac{Q}{c}\vec{A}$$

$$\vec{J} = \frac{Q}{m_q} n_Q \vec{p} - \frac{Q^2}{m_Q c} n_Q \vec{A}$$

Superconducting state is a coherent state of charge carriers with vanishing canonical momentum

$\vec{p} = 0$ assumption of London

This now has a firm QM foundation involving phase of wavefunction

Effective current density in superconductor is

$$\vec{J} = -\frac{Q^2}{m_Q c} n_Q \vec{A}$$

Insert this into Lorenz-gauge wave eq. for $\vec{A}$

$$\left[\nabla^2 - \partial_0^2\right] A = -\frac{4\pi}{c}\vec{J}$$

$$= \left[\frac{4\pi Q^2}{m_Q c^2} n_Q\right] \cancel{\nabla} A$$

Compare to Proca form

$$\left[\nabla^2 - \partial_0^2\right]\vec{A} - \mu^2 \vec{A} = 0$$

$$\mu^2 = \frac{4\pi Q^2 n_Q}{m_Q c^2}$$

Inside a superconductor the photon accuries an effective mass do to its interaction with the charge carriers.

As a result the $\vec{B}$ field will decay away exp. with a London penetration depth $\lambda_L = \mu^{-1}$

$$\lambda_L = \sqrt{\frac{m_Q c^2}{4\pi Q^2 n_Q}}$$

Effective photon mass $= \hbar/(\lambda_L c) = m_{eff}$

Charge carriers shurely related to electrons

$$Q \sim e$$

$$m_Q \sim m_e$$

$$n_Q \sim \frac{1}{a_0^3}$$

$$m_{eff} c^2 = \left[\left|\frac{Q}{e}\right| \sqrt{\frac{4\pi \, n_Q \, a_0^3}{m_Q / m_e}} \right] \frac{e^2}{a_0}$$

Square brackets is assumed order unity

Effective mass is order of Rydberg energy $\sim$ few eV.

Experiment and theory show charge carriers are pairs of electrons

$$Q = -2e$$

$$m_Q = 2m_e$$

$$n_Q = n_{eff}/2$$

n_{eff} = effective # of electrons contributing to current

Only electrons near Fermi surface form Cooper pairs.

In BCS theory of superconductivity

J. Bardeen, L. N. Cooper, J.R. Schrieffer
Phys. Rev. 108, 1175 (1957)

$$\frac{n_{eff}}{2} = \frac{2}{3} E_F \; N(0)$$

$N(0)$ density of states (# of states per unit energy of one spin state) at Fermi surface

E_F = Fermi energy of valance band

$$\lambda_L \simeq O\left(4 \times 10^{-6} \text{ cm}\right)$$

Next week

Review

Some discussion of P506
then most of time on P507

Work some example problems

Formula sheet for final

Note P507 exam
Thursday May 5 2:45 pm

P512 exam will move to Wed.
May 4 at 2:45 pm

Review (Part 0) P 506

Chap. 1 Intro to electrostatics

Coulomb's law $F = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x_1 - x_2}}{|x_1 - x_2|^3}$

Electric field $\vec{E} = \vec{F}/q$

Gauss law $\nabla \cdot E = \rho/\epsilon_0$

$$E = -\nabla \Phi \quad , \qquad \Phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')}{|x - x'|} d^3x'$$

Green's fnc.

$$\nabla_x^2 \, G(x, x') = -4\pi \, \delta(\vec{x - x'})$$

$$G = \frac{1}{|x - x'|} + f(x, x') \quad , \quad \nabla_x^2 f = 0$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(x') \, G(x, \vec{x}') \, d^3x'$$

$$+ \frac{1}{4\pi} \oint_S \left[G(x, x') \frac{\partial \Phi}{\partial n'} - \Phi(x') \frac{\partial G}{\partial n'} \right] da$$

S = surface bounding V

Dirichlet b.c. Φ specified on S

Neumann b.c $\frac{\partial \Phi}{\partial n'}$ specified on S

Chap. 2 Boundary-Value Problems in Electrostatics I

1) Method of images
2) Orthogonal function expansions
3) Separation of Variables

Chap. 3 B. V. Problems II

Laplace eq. in Spherical coord.
Legendre eq. / legendre polynomials

General solution of laplace eq. $\nabla^2 \Phi = 0$
with azimuthal sym.

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\frac{1}{|x - x'|} = \sum_{l=0}^{\infty} \left(\frac{r_<^l}{r_>^{l+1}} \right) P_l(\cos\theta)$$

Spherical addition thm.

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)$$

$$\cos\gamma = \hat{x} \cdot \hat{x}'$$

Chap. 4 Multipoles, Macroscopic media

assume $\rho(x')$ localized in $r' < \Lambda$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(x')}{|x - x'|}$$

expand $1/|x - x'|$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta\phi)}{r^{l+1}}$$

$$q_{lm} = \int d^3x' \, \rho(\vec{x}') \, r'^l \, Y^*_{lm}(\theta'\phi')$$

$q_{1m} \leftrightarrow$ electric dipole moment

$q_{2m} \leftrightarrow$ electric quadrupole moments

$$Q_{ij} = \int d^3x' \, \rho(x') \, (3x_i' x_j' - \delta_{ij} r'^2)$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \, Q_{33} \quad \text{etc.}$$

<u>Electrostatics of ponderable media</u>

$$\vec{D} = \vec{P} + \epsilon_0 \vec{E}$$

$$P(x) = \sum_i N_i \langle p_i \rangle$$

N_i density of molecules of type i

p_i = electric dipole moment of i^{th} type of molecule

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

χ_e = electric susceptibility

Chap. 5 Magnetostatics

Biot Savart Law

$$dB = kI \frac{d\ell \wedge \vec{x}}{x^3}$$

Integrate ove $J(x)\, d^3x$

$$B(x) = \frac{\mu_0}{4\pi} \int J(x') \wedge \frac{\vec{x - x'}}{|\vec{x - x'}|^3} d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla \wedge \int \frac{J(x')}{|x - x'|} d^3x'$$

$\Rightarrow \nabla \cdot B = 0$

$\nabla \times B = \mu_0 J$ Ampere's law

Vector Pot.

$B = \nabla \times A$, $A = \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x-x'|} d^3x'$

magnetic moment $\vec{m} = \frac{1}{2} \int \vec{x}' \wedge J(x') d^3x'$

$A = \frac{\mu_0}{4\pi} \frac{m \wedge x}{x^3}$ $B = \left(\frac{\mu_0}{4\pi}\right) \frac{3\hat{n}\, \hat{n}\cdot\vec{m} - \vec{m}}{|x|^3}$

magnetic dipole field

$H = \frac{1}{\mu_0} B - M$

M = magnetization = $N \langle m \rangle$

N density of molecules

$\vec{m}$ magnetic moment of a molecule

H = Magnetic field

B = magnetic induction

Constitutive relation between H, B

example $B = f(H)$

for isotropic linear material

$$B = \mu H \qquad \mu = \text{magnetic permeability}$$

Review
Boundary conditions for B, H, E, D
at interface between media

Chap. 6 Maxwell Equations

$\nabla \cdot D = \rho$ Coulomb's law

$\nabla \times H = J + \partial D/\partial t$ Ampere's law with displacement current

$\nabla \times E + \dfrac{\partial B}{\partial t} = 0$ Faraday's law

$\nabla \cdot B = 0$ No magnetic charge

Displacement current insures

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \qquad \text{continuity eq.}$$

$$B = \nabla \times A \ , \quad E = -\frac{\partial A}{\partial t} - \vec{\nabla} \Phi$$

Wave eq.

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_0 J_t$$

in Coulomb gauge $\nabla \cdot A = 0$

$$J = J_\ell + J_t \ , \quad \nabla \cdot J_t = \nabla \wedge J_\ell = 0$$

Green's func. for wave eq.

$$\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) G^{\pm}(xt\,;\,x't')$$
$$= -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

$$G^{\pm} = \frac{\delta\left[t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right]}{|\vec{x}-\vec{x}'|}$$

Retarded G^+ and advanced G^- depend on b.c. in time

With G^+ fields at time t depend on ρ, J at retarded time

$$t' = t - |\vec{x}-x'|/c$$

Poynting thm and conservation of Energy / momentum

$S = E \wedge H$ Energy current

$$\nabla\cdot S + \frac{\partial u}{\partial t} = -J\cdot E$$

$$u = \tfrac{1}{2}(E\cdot D + B\cdot H)$$

energy density in EM fields