Lec. 25 Review

Last time photon mass term

$$\mathcal{L}_{\text{photon mass term}} = -\frac{1}{16\pi} F_{\alpha \beta} F^{\alpha \beta} + \frac{\alpha}{8\pi} \mathcal{A}_\alpha A^\alpha - \frac{1}{c^2} \mathcal{A}_\alpha A^\alpha$$

Euler Lag. eq. assuming $2\pi A_d = 0$

$$\nabla A^\alpha + m^2 A^\alpha = \frac{4\pi}{c} J^\alpha$$

Static limit

$$\nabla^2 A^\alpha - m^2 A^\alpha = -\frac{4\pi}{c} J^\alpha$$

For a charge at rest

$$A^\alpha = \Phi = q e^{-\frac{er}{r}}$$

Falls exp.

In superconducting medium photon has an effective mass because of its interactions with charge carriers.

This causes expected magnetic field to be present from screening distance.

\begin{align*}
\text{Superconductor} &\quad \text{Normal region} \\
B=0 &\quad B\neq 0, M=0 \\
\mu \neq 0 &\quad B\text{ decays exp. at surface}
\end{align*}
If source is a pt. charge at \( \text{comp.} \) of \( \text{comp.} \), only time comp. of \( A_0 \) is nonzero.

\[
A_0 = \Phi \\
\Phi(x) = q \frac{e^{-px}}{r}
\]

Characteristic feature: fall off of photon of mass \( \mu \).

This alters the character of the earth's magnetic field enough to allow mass from geomagnetic data.

Effective "photon" mass in superconductor.

London penetration depth.

London theory of electromagnetic behavior of superconductors.

Explains Meissner effect (1933) of expulsion of magnetic field from interior of superconductor.

Phenomenological discussion of current in superconductor.

Assume

\[
\vec{J} = \vec{Q} \cdot n \cdot q \cdot \vec{v}
\]

\( Q \) = charge of current carriers

\( n \cdot q \) = density of carriers
In presence of E+M fields

\[ \vec{P} = m \vec{v} + \frac{Q}{c} \vec{A} \]

\[ \vec{J} = \frac{Q}{m} \frac{n Q}{c} \vec{P} - \frac{Q^2}{m c} n Q \vec{A} \]

Superconducting state is a coherent state of charge carriers with vanishing canonical momentum \( \vec{P} = 0 \) assumption of London

This now has a firm QM foundation

Involving phase of wave function

Effective current density in superconductors is

\[ \vec{J} = -\frac{Q^2}{m c} n Q \vec{A} \]

Insert this into a Lorenz-gauge wave eq.

\[ \left[ \nabla^2 - \frac{1}{c^2} \right] A = -\frac{4\pi i}{c} \frac{\vec{J}}{m c^2} \]

Form \( \frac{4\pi i Q^2}{m c^2} n Q \)

Compare to Proca form

\[ \left[ \nabla^2 - \frac{1}{c^2} \right] \vec{A} - \frac{\mu^2}{m^2} \vec{A} = 0 \]

\[ \mu^2 = -\frac{4\pi i Q^2}{m c^2} n Q \]
Inside a superconductor, the photon acquires an effective mass due to its interaction with the charge carriers. As a result, the $B$ field will decay away, with a London penetration depth $\lambda_L = \hbar^2 / \pi m e^2$

$$\lambda_L = \sqrt{\frac{m e^2}{4\pi Q^2 n Q}}$$

Effective photon mass $= \hbar / (\lambda_L) = m_{eff}$

Charge carriers are likely related to $Q = e$

$$m_Q \approx m_e$$

$$n_Q \approx \frac{1}{6.3 \times 10^{-3}}$$

$$m_{eff} c^2 = \left[ \frac{|Q|}{e} \sqrt{\frac{4\pi n_Q a_0}{m_Q \times m_e}} \right] \frac{e^2}{a_0}$$

Square brackets is assumed order unity

Effective mass energy $\sim$ few $\text{eV}$
Experiment and theory show charge carriers are pairs of electrons

\[ Q = -2e \]
\[ M_Q = 2me \]
\[ n_Q = \frac{n_{eff}}{2} \]
\[ n_{eff} = \text{effective # of electrons contributing to current} \]

Only electrons near Fermi surface form Cooper pairs.

In BCS theory of superconductivity, J. Bardeen, L. N. Cooper, J. R. Schrieffer


\[ \frac{n_{eff}}{2} = \frac{2}{3} E_f N(0) \]

\[ N(0) \text{ density of states / # of states per unit energy at Fermi surface} \]

\[ E_f = \text{Fermi energy of valence band} \]

\[ L \approx 0 \left( 4 \times 10^{-6} \text{ cm} \right) \]
Next Week

Review

Some discussion of P 506
then most of time on P 507
Work some example problems

Formula sheet for final

Note P 507 exam Thursday May 5 2:45 pm
P 512 exam will move to Wed. May 4 at 2:45 pm
Chap. 1 Intro to electostatics

Coulomb's law \[ F = \frac{q_1 q_2}{4\pi \varepsilon_0} \frac{x_1 - x_2}{(x_1 - x_2)^3} \]

Electric field \[ \vec{E} = \frac{q}{r^2} \]

Gauss law \[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

\[ E = -\nabla \Phi \]

\[ \Phi(x) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(x')}{|x - x'|} \, dx' \]

Green's func.

\[ \nabla_x^2 G(x, x') = -4\pi \delta(\vec{x} - \vec{x}') \]

\[ G = \frac{1}{|x - x'|} + \int G(x, x') \, \nabla_x^2 f = 0 \]

\[ \Phi(x) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(x') G(x, x')}{|x - x'|} \, dx' \]

\[ + \frac{1}{4\pi} \oint \left[ G(x, x') \frac{\partial \Phi}{\partial n'} - \Phi(x') \frac{\partial G}{\partial n} \right] \, da \]

\( S = \text{Surface bounding } V \)

Dirichlet b.c. \[ \Phi \text{ specified on } S \]

Neumann b.c. \[ \frac{\partial \Phi}{\partial n} \text{ specified on } S \]
Chap. 2  Boundary-Value Problems in Electrostatics

1) Method of images
2) Orthogonal function expansions
3) Separation of variables

Chap. 3  B.V. Problems II

Laplace eq. in spherical coord.
Legendre eq. / Legendre poly. normals

General solution of Laplace eq. \( \nabla^2 \phi = 0 \)
with azimuthal sym.

\[
\phi(r, \theta) = \sum_{l=0}^{\infty} \left( a_l \frac{r^l}{r^{l+1}} + \frac{b_l}{r^{l+1}} \right) \phi_l(\cos \theta)
\]

\[
\frac{1}{|x-x'|} = \sum_{l=0}^{\infty} \left( \frac{r^l}{r^{l+1}} \right) \phi_l(\cos \theta)
\]

Spherical addition thm.

\[
\phi_l(\cos \theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\theta, \phi) Y_l^m(\theta, \phi')
\]

\[
\cos \gamma = \mathbf{x} \cdot \mathbf{x}'
\]

Chap. 4  Multipoles
Macroscopic media
Assume \( \rho(x') \) localized in \( r' < \Lambda \)

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(x')}{|x-x'|}
\]

Expand \( \frac{1}{|x-x'|} \)
\[ \Phi(x) = \frac{1}{4\pi e_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{l+1} \bar{q}_{lm} Y_{lm}(\theta \phi) \]

\[ q_{lm} = \int d^3x' \rho(x') r^{l+1} Y^*_{lm}(\theta' \phi') \]

\[ q_{lm} \leftrightarrow \text{electric dipole moment} \]

\[ q_{am} \leftrightarrow \text{electric quadrupole moments} \]

\[ Q_{ij} = \int d^3x' \rho(x') \left( 3x_i' x_j' - \delta_{ij} r^2 \right) \]

\[ q_{20} = \frac{1}{2} \sqrt{\frac{5}{4\pi}} Q_{33} \text{ etc.} \]

**Electrostatics of ponderable media**

\[ \mathbf{D} = \mathbf{P} + \varepsilon_0 \mathbf{E} \]

\[ \rho(x) = \sum_i N_i \mathbf{p}_i \]

- \( \rho(x) \) is the density of molecules of type \( i \)
- \( N_i \) is the number of molecules of type \( i \)
- \( \mathbf{p}_i \) is the electric dipole moment of the molecule of type \( i \)

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \]

- \( \chi_e \) is the electric susceptibility
Chap. 5 Magnetostatics

Biot-Savart Law

\[ dB = \kappa I \frac{dl \times A}{x^3} \]

Integrate over \( J(x') \, d^3 x' \)

\[ B(x) = \frac{\mu_0}{4\pi} \nabla \times \sum \frac{J(x')}{|x-x'|^3} \, d^3 x' \]

\[ = \frac{\mu_0}{4\pi} \nabla \times \sum \frac{J(x')}{|x-x'|} \, d^3 x' \]

\( \Rightarrow \nabla \cdot B = 0 \)

\[ \nabla \times B = \mu_0 J \]

Amperes law

Vector Potential

\[ B = \nabla \times A \]

\[ A = \frac{\mu_0}{4\pi} \sum \frac{J(x')}{|x-x'|} \, d^3 x' \]

magnetic moment

\[ \vec{m} = \frac{1}{2} \sum \vec{x}' \times J(x') \]

\[ \vec{B} = \frac{\mu_0}{4\pi} \frac{3 \hat{n} \cdot \vec{m} - \vec{m}}{|x|^3} \]

magnetic dipole field

\[ H = \frac{1}{\mu_0} B - M \]

\[ M = \text{magnetization} = N \langle m \rangle \]

\( M \) = magnetic field

\( B \) = magnetic induction

\( \vec{m} \) = magnetic moment of a molecule
Constitutive relation between $H$, $B$

example $B = f(H)$

for isotropic linear material

$B = \mu H$

$\mu =$ magnetic permeability

Review

boundary conditions for media: $E$, $D$

Chap. 6 Maxwell Equations

$\nabla \cdot D = 0$

Coulomb's law

$\nabla \times H = J + \frac{dM}{dt}$

Ampere's law

with displacement current

$\nabla \times E + \frac{dB}{dt} = 0$

Faraday's law

$\nabla \cdot B = 0$

No magnetic charge

Displacement current: $\nabla \cdot J + \frac{d\Psi}{dt} = 0$

continuity eq.

$B = \nabla \times A$

$E = -\frac{\partial A}{\partial t} - \vec{\nabla} \Phi$

Wave eq.

$\nabla^2 - \frac{1}{c^2} \frac{d^2A}{dt^2} = -M_0 \frac{J}{c}$

in Coulomb gauge $\nabla \cdot A = 0$

$J = J + J_d$

$\nabla \cdot J_t = \nabla \cdot J_d = 0$
Green's func. for wave eq.

\[
\left( \nabla_x^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^+ (x,t; x', t') = -\delta(\vec{x} - \vec{x}') \delta(t - t')
\]

\[
G^+ = \frac{\delta(t - t' - \frac{\mid \vec{x} - \vec{x}' \mid}{c})}{\mid \vec{x} - \vec{x}' \mid}
\]

Retarded \( G^+ \) and advanced \( G^- \) depend on b.c. in time.

With \( G^+ \) fields depend on \( \vec{p} \) \( \mathcal{J}^+ \) at time \( t \) at retarded time \( t' = t - \frac{\mid \vec{x} - \vec{x} \mid}{c} \).

Pointing them and conservation of Energy / Momentum:

\[
S = E \times H
\]

\[
\nabla \cdot S + \frac{\partial \mathcal{J}^+}{\partial t} = -\mathcal{J} \cdot E
\]

\[
\rho = \frac{1}{2} \left( E \cdot \mathcal{D} + B \cdot \mathcal{H} \right)
\]

\textit{Energy density in EM fields}.