Lec. 24 Lagrangian for E+M fields

Last time Darwin Lagrangian

Interaction to order $(V/c)^2$

Start from

$$L = -m^2 \left(1 - \frac{V^2}{c^2}\right) + e \mathbf{A} \cdot \mathbf{E} - e\Phi$$

and calculate $\mathbf{A}$ from 2nd particle to order $V/c$

$$A_{12} \approx \frac{1}{c} \int J_t(x') d^3x' \frac{1}{|x_1 - x'_2|}$$

$$J_t(x') = q_2 v_2 \delta(x' - x_2) - \frac{q_2}{4\pi} \nabla \cdot \frac{\mathbf{v}_2 \cdot (x_1 - x_2)}{|x_1 - x_2|^3}$$

Plug in and do integral

$$A_{12} = \frac{q_2}{c} \left( \frac{\mathbf{v}_2}{r} - \frac{1}{2} \nabla \left( \frac{\mathbf{v}_2 \cdot \mathbf{v}_2}{r} \right) \right)$$

$$A_{12} = \frac{q_2}{ac r} \left( \mathbf{v}_2 + \frac{\mathbf{r} \cdot \mathbf{v}_2}{r^2} \right)$$

Lagrangian:

$$L_{\text{Darwin}} = \frac{1}{2} \sum_i m_i v_i^2 + \frac{1}{8c^2} \sum_i m_i \frac{v_i^4}{2} - \frac{1}{2} \sum q_i q_j \sum_{i \neq j} \left[ \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{r}_{ij} \mathbf{v}_j \cdot \mathbf{r}_{ij} \right]$$
Lagrangian for E+M Field

Field theory

In Lagrangian dynamics one has a finite # of degrees of freedom
For example, 3N, where N is the # of particles

$q_i$, $i = 1, \ldots, 3N$

Now replace this with one coordinate for every point in space

$i \rightarrow x^k$

Field theory has an infinite # of degrees of freedom.

$q_i \rightarrow \phi^k(x)$

$L = L(q^i, \dot{q}^i; x) \Rightarrow \int L(\phi, \dot{\phi}) dx$

$L = \text{Lagrangian density}$

$A = \int d^3x \int dt \ L = \frac{1}{c} \int d^4x$

Action is invariant if $L$ is a Lorentz scalar.

Euler - Lagrange eqs

$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$
\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\partial \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k} \]

The Lagrangian for the free E + M field is expected to be at least quadratic in velocities \( \mathbf{A}' \rightarrow \mathbf{A} \) or in general \( F^{\mu\nu} \tilde{F}_{\mu\nu} \).

Only Lorentz invariant quadratic form is

\[ F^{\mu\nu} F_{\mu\nu} \]

Note could also have dual

\[ \tilde{\mathcal{A}}^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\nu\rho} \]

\[ \varepsilon = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \]

Where

\[ F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \]

\[ \tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & B_y & B_x & 0 \end{bmatrix} \]
\[ F_{\alpha \beta} F^{\alpha \beta} = \text{tr} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & -B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \]

\[ = -E_x^2 - E_y^2 - E_z^2 + E_x^2 - B_z^2 - B_y^2 + E_y^2 - B_z^2 - B_x^2 + E_z^2 - B_y^2 - B_x^2 \]

\[ = 2 (E^2 - \mathbf{B} \cdot \mathbf{B}) \]

The combination \( F_{\alpha \beta} F^{\alpha \beta} \) will involve \( \mathbf{E} \cdot \mathbf{B} \).

This is rotationally invariant but is odd under parity (space inversion).

Under parity \( \mathbf{E} \rightarrow -\mathbf{E} \), \( \mathbf{B} \rightarrow \mathbf{B} \), so \( \mathbf{E} \cdot \mathbf{B} \) is even under parity but \( E^2 \) is odd.

For the interaction term we could guess \( J_\alpha A^\alpha \), which is a Lorentz scalar. Start from

\[ L_{\text{int}} = -\frac{e}{\gamma c} A^\alpha J_\alpha \]

and replace \( \frac{e}{\gamma c} v \) particle velocity with \( J_\alpha \).

\[ L = \frac{-1}{16 \pi} F_{\alpha \beta} F^{\alpha \beta} - \frac{1}{c} J_\alpha A^\alpha \]
The sign and coeff. of $J \alpha \phi$ with $\bar{A}$ term chosen to agree with coeff. of $F_{\alpha \beta}$ term to insure correct Maxwell eqs.

Calculating $\frac{\partial \phi}{\partial (\partial \phi \phi)}$ can be a little tricky.

Write

$$\phi = -\frac{1}{16\pi} \int g_{\alpha \mu} g_{\nu \sigma} \left( \delta A^\alpha \delta A^\mu - \delta A^\mu \delta A^\alpha \right)$$

$$\frac{\partial \phi}{\partial (\partial \phi \phi)}$$

$$\frac{\partial \phi}{\partial (\partial \phi \phi)} = -\frac{1}{16\pi} \int g_{\alpha \mu} g_{\nu \sigma} \left( \delta A^\alpha \delta A^\mu - \delta A^\mu \delta A^\alpha \right)$$

$$= -\frac{1}{16\pi} \int \left[ F_{\beta \alpha} - F_{F \beta \alpha} \right]$$

$$= -\frac{1}{4\pi} F_{\beta \alpha} = \frac{1}{4\pi} F_{\alpha \beta}$$

$$\frac{\partial \phi}{\partial (\partial \phi \phi)}$$

$$\frac{\partial \phi}{\partial (\partial \phi \phi)} = -\frac{1}{c} \frac{\partial \phi}{\partial \phi}$$

Covariant form of inhomogenous Maxwell eqs. (11.141)

Note since $F_{\beta \alpha} = -F_{\alpha \beta}$

$$\frac{1}{4\pi} \frac{\partial \phi}{\partial (\partial \phi \phi)} = \frac{1}{c} \frac{\partial \phi}{\partial \phi}$$

$$\frac{1}{4\pi} F_{\beta \alpha} = \frac{1}{c} \frac{\partial \phi}{\partial \phi}$$
Proca Lagrangian and Photon Mass

Assume photon had mass $\mu$

$$L = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J A_\alpha$$

Note extra term is a Lorentz scalar

Maxwell eqs get an extra term for

$$\frac{\partial R_{\text{Proca}}}{\partial A_\alpha} = \frac{\mu^2}{4\pi} A_\alpha - \frac{1}{c} J A_\alpha$$

$$\frac{1}{4\pi} \int d^3 F_{\alpha\beta} + \frac{\mu^2}{4\pi} A_\alpha = \frac{1}{c} J A_\alpha$$

or

$$\frac{1}{4\pi} \int d^3 F_{\alpha\beta} + \mu^2 A_\alpha = \frac{4\pi}{c} J A_\alpha$$

Now $A_\alpha$ is physical and we

$$\partial A_\alpha = 0$$

so that current is conserved $\partial A_\alpha = 0$

(No longer have gauge invariance)

$$\partial_\beta F_{\alpha\beta} = \partial_\beta \partial_\beta A_\alpha - \frac{1}{c} \partial_\beta J A_{\beta\alpha} = \square A_\alpha$$

$$\square A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J A_\alpha$$

In static limit

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J A_\alpha$$