Lec. 24 Lagrangian for E+M fields

Last time <u>Darwin Lagrangian</u>

Interaction to order $(v/c)^2$

Start from $L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\vec{v}\cdot\vec{A} - e\Phi$

and calculate $\vec{A}$ from 2nd particle to order v/c

$$A_{12} \cong \frac{1}{c}\int \frac{J_t(\vec{x}')\, d^3x'}{|\vec{x}_1 - \vec{x}'|}$$

$$J_t(x') = q_2 v_2\, \delta(\vec{x}' - \vec{x}_2) - \frac{q_2}{4\pi}\nabla' \frac{v_2\cdot(x'-x_2)}{|x'-x_2|^3}$$

Plug in and do integral

$$\vec{A}_{12} = \frac{q_2}{c}\left(\frac{\vec{v}_2}{r} - \frac{1}{2}\nabla_r\left(\frac{\vec{v}_2\cdot\vec{r}}{r}\right)\right)$$

$$\vec{A}_{12} = \frac{q_2}{2cr}\left(\vec{v}_2 + \frac{\vec{r}\;\vec{v}_2\cdot\vec{r}}{r^2}\right)$$

$$L_{int} = \frac{q_1 q_2}{r}\left\{-1 + \frac{1}{2c^2}\left[\vec{v}_1\cdot\vec{v}_2 + \frac{\vec{v}_1\cdot\vec{r}\;\vec{v}_2\cdot\vec{r}}{r^2}\right]\right\}$$

$$\boxed{L_{Darwin} = \frac{1}{2}\sum_i m_i v_i^2 + \frac{1}{8c^2}\sum_i m_i v_i^4 - \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}} + \frac{1}{4c^2}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}}\left[\vec{v}_i\cdot\vec{v}_j + \vec{v}_i\cdot\hat{r}_{ij}\;\vec{v}_j\cdot\hat{r}_{ij}\right]}$$

Lagrangian for E+M field

Field theory

In Lagrangian dynamics one has a finite # of degrees of freedom for example 3N where N is the # of particles

$$q_i \qquad i = 1, \dots, 3N$$

Now replace this with one coordinate for every point in space

$$i \rightarrow x^\alpha$$

field theory has an infinite # of degrees of freedom.

$$q_i \rightarrow \phi_k(x) \qquad k = \text{field}$$

$$\dot{q}_i \rightarrow \partial^\alpha \phi_k(x)$$

$$L = L(\{q_i, \dot{q}_i\}) \rightarrow \int \mathcal{L}(\phi_k, \partial^\alpha \phi_k)\, d^3x$$

$\mathcal{L}$ = Lagrangian density

$$A = \int d^3x \int dt\ \mathcal{L} = \frac{1}{c} \int \mathcal{L}\, d^4x$$

action is invariant if $\mathcal{L}$ is a Lorentz scalar.

Euler - Lagrange eqs

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

become

$$\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k}$$

The Lagrangian for the ~~fra~~ E+M field is expected to be ~~at least~~ quadratic in velocities

$\partial^0 A^1 \rightarrow$ or in general $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

Only Lorentz invariant quadratic form is

$$F^{\alpha\beta} F_{\alpha\beta}$$

Note could also have daul

$$\mathcal{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\mu\delta} F_{\mu\delta}$$

$$= \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

where

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\alpha\beta}F^{\alpha\beta} = \mathrm{tr}\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$= \vec{E}\cdot\vec{E} + E_x^2 - B_z^2 - B_y^2 + E_y^2 - B_z^2 - B_x^2 + E_z^2 - B_y^2 - B_x^2$$

$$= 2\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right)$$

The combination $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ will involve $\vec{E}\cdot\vec{B}$

This is rotationaly invariant but is odd under parity (space inversion)

Under parity $\vec{E} \to -\vec{E}$

$\vec{B} \to +\vec{B}$

so $\vec{E}^2$ and $\vec{B}^2$ is even under parity but $\vec{E}\cdot\vec{B}$ is odd

For the interaction term we could guess

$$J_\alpha A^\alpha$$

which is a Lorentz scalar. Start from

$$L_{int} = -\frac{e}{\gamma c} U_\alpha A^\alpha$$

and replace $\frac{eU_\alpha}{\gamma}$ particle velocity with J_α

$$\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha$$

The sign and coef. of $J_\alpha A^\alpha$ term chosen to agree with L_{int}. Check coef. of $F_{\alpha\beta} F^{\alpha\beta}$ term to insure correct Maxwell eqs.

Calculating $\frac{\partial}{\partial(\partial^\beta A^\alpha)} \mathcal{L}$ can be a little tricky

Write

$$\mathcal{L} = -\frac{1}{16\pi} g_{\lambda\mu}\, g_{\nu\sigma} \left(\partial^\mu A^\sigma - \partial^\sigma A^\mu\right)\left(\partial^\lambda A^\nu - \partial^\nu A^\lambda\right) - \frac{1}{c} J_\alpha A^\alpha$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{16\pi} g_{\lambda\mu}\, g_{\nu\sigma} \left\{ \delta^\mu_\beta \delta^\sigma_\alpha F^{\lambda\nu} - \delta^\sigma_\beta \delta^\mu_\alpha F^{\lambda\nu} + \left(\delta^\lambda_\beta \delta^\nu_\alpha - \delta^\nu_\beta \delta^\lambda_\alpha\right) F^{\mu\sigma} \right\}$$

$$= -\frac{1}{16\pi}\left[F_{\beta\alpha} - F_{\alpha\beta} + F_{\beta\alpha} - F_{\alpha\beta} \right]$$

$$= -\frac{1}{4\pi} F_{\beta\alpha} = \frac{1}{4\pi} F_{\alpha\beta}$$

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha \Rightarrow \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0$$

$$\boxed{\frac{1}{4\pi} \partial^\beta F_{\beta\alpha} = \frac{1}{c} J_\alpha}$$

Covariant form of inhomogeneous Maxwell eqs (11.141)

Note since $F_{\beta\alpha} = -F_{\alpha\beta}$ the current is conserved

$$\frac{1}{4\pi} \partial^\alpha \partial^\beta F_{\beta\alpha} = \frac{1}{c} \partial^\alpha J_\alpha = 0$$

Proca Lagrangian and Photon Mass

assume photon had mass μ

$$\mathcal{L}_{Proca} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha$$

~~Now~~ Note extra term is a Lorentz scalar

Maxwell eqs get an extra term from

$$\frac{\partial \mathcal{L}_{Proca}}{\partial A^\alpha} = \frac{\mu^2}{4\pi} A_\alpha - \frac{1}{c} J_\alpha$$

$$\frac{1}{4\pi} \partial^\beta F_{\beta\alpha} + \frac{\mu^2}{4\pi} A_\alpha = \frac{1}{c} J_\alpha$$

or

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha$$

Now A_α is physical and we require

$$\partial^\alpha A_\alpha = 0 .$$

so that current is conserved $\partial^\alpha J_\alpha = 0$

(No longer have gauge invariance)

$$\partial^\beta F_{\beta\alpha} = \partial^\beta \partial_\beta A_\alpha - \underbrace{\partial^\beta \partial_\alpha A_\beta}_{=0} = \Box A_\alpha$$

$$\Box A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha$$

In static limit

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha$$