Lec. 21 4/12/11

Relativistic Eq. for Spin

Generalize spin to a 4 vector

Let S^α denote components of spin in inertial frame K

Time component of S'^0 in rest frame K'

$$S'^0 = \gamma(S^0 - \beta \cdot S) = \frac{1}{c} U_\alpha S^\alpha$$

Choose $S'^\alpha = (0, \vec{s})$

Impose constraint $U_\alpha S^\alpha = 0$

In an inertial frame where particle has velocity $c\vec{\beta}$

$$S_0 = \vec{\beta} \cdot \vec{S}$$

Now a vector under a L.T. (11.19)

$$\vec{x}' = \vec{x} + \left(\frac{\gamma - 1}{\beta^2}\right) \beta \cdot x \beta - \gamma \beta x_0$$

So

$$\vec{s} = \vec{S} + \left(\frac{\gamma - 1}{\beta^2}\right) \beta \cdot \vec{S} \beta - \gamma \beta (\vec{\beta} \cdot \vec{S})$$

particles spin in its rest frame

$$\cancel{\vec{s} = \vec{S} - \beta \cdot \vec{S} \beta}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad \frac{\gamma(1-\beta^2) - 1}{\beta^2}$$

$$1 - \beta^2 = \frac{1}{\gamma^2} \qquad 1 - \frac{1}{\gamma^2} = \beta^2$$

$$= \frac{1/\gamma - 1}{\beta^2} = \frac{1 - \gamma}{\gamma \beta^2}$$

$$= \frac{1/\gamma - 1}{1 - 1/\gamma^2} = \frac{\gamma - \gamma^2}{\gamma^2 - 1} = \frac{\gamma(1-\gamma)}{(\gamma-1)(\gamma+1)} = -\frac{\gamma}{1+\gamma}$$

$$\vec{s} = \vec{S} \cancel{\gamma} - \left(\frac{\gamma}{\gamma+1}\right)(\beta\cdot\vec{S})\vec{\beta}$$

Inverse from (11.19) with $s_0 = 0$ in rest frame

$$\vec{S} = \vec{s} + \left(\frac{\gamma-1}{\beta^2}\right)\vec{\beta}\cdot\vec{s}\,\vec{\beta} \qquad \beta^2 = 1 - \frac{1}{\gamma^2}$$

$$= \vec{s} + \left(\frac{\gamma-1}{1-1/\gamma^2}\right)\beta\cdot s\,\beta = \vec{s} + \frac{\gamma^2(\gamma-1)}{\gamma^2-1}\beta\cdot s\,\beta$$

$$\vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}(\vec{\beta}\cdot\vec{s})\vec{\beta}$$

$$S_0 = \gamma(s_0 + \beta\cdot\vec{s}) = \gamma(\vec{\beta}\cdot\vec{s})$$

If you know 3-vector spin $\vec{s}$ in particles rest frame $\Rightarrow$ determines components of S^α in any inertial frame.

Consider nonrel. eq. of motion for spin $\vec{s}$ for a particle with a magnetic moment

$$\vec{\mu} = g\frac{e}{2mc}\vec{s}$$

The g factor describes structure of particle. The magnetic moment should point in the direction $\vec{s}$. If a particle with charge e and mass m had orbital angular momentum L the magnetic dipole moment is

$$\vec{\mu} = \frac{e}{2mc}\vec{L}$$

The Dirac eq. implies $g=2$ for a point electron or proton.

Nonrel. Torque eq.

$$\frac{d\vec{s}}{dt'} = \frac{ge}{2mc}\,\vec{s} \wedge B'$$

Prime denotes quantities in particle's rest frame.

We would like a relativistic generalization of this eq.

Try generalizing L.H.S to $\frac{dS^\alpha}{d\tau}$

Need RHS as a 4 vector. Expect it to be linear in $F^{\alpha\beta}$, can also involve U^α and $\frac{dU^\beta}{d\tau}$

Note Lorentz force Law

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta$$

$$\Rightarrow \quad \frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$$

We also expect RHS to be linear in S^β thus

$$\frac{dS^\alpha}{d\tau} = A_1 F^{\alpha\beta}S_\beta + \frac{A_2}{c^2}\left(S_\lambda F^{\lambda\mu}U_\mu\right)U^\alpha + \frac{A_3}{c^2}S_\beta\left(\frac{dU^\beta}{d\tau}\right)U^\alpha$$

Other possibilities such as $F^{\alpha\beta}U_\beta S_\lambda U^\lambda$, $U_\lambda F^{\lambda\mu}U_\mu S^\alpha$, $S_\lambda F^{\lambda\mu}U_\mu \frac{dU^\alpha}{d\tau}$ either vanish, are higher order in $F^{\alpha\beta}$ or reduce to multiples of above.

The constraint

$$U_\alpha S^\alpha = 0$$

should be true for all time

$$\frac{d}{d\tau}\left(U_\alpha S^\alpha\right) = S^\alpha \frac{dU_\alpha}{d\tau} + U_\alpha \frac{dS^\alpha}{d\tau} = 0$$

$$U_\alpha \frac{dS^\alpha}{d\tau} = A_1 U_\alpha F^{\alpha\beta} S_\beta + \frac{A_2}{c^2} S_\lambda F^{\lambda\mu} U_\mu U_\alpha U^\alpha + \frac{A_3}{c^2} S_\beta \frac{dU^\beta}{d\tau} U_\alpha U^\alpha$$

$$U_\alpha U^\alpha = c^2$$

$$U_\alpha \frac{dS^\alpha}{d\tau} = A_1 U_\alpha F^{\alpha\beta} S_\beta + A_2 S_\lambda F^{\lambda\mu} U_\mu + A_3 S_\beta \frac{dU^\beta}{d\tau} = - S^\alpha \frac{dU_\alpha}{d\tau}$$

Note $F^{\lambda\mu} = F^{\mu\lambda}$

$$\Rightarrow \quad (A_1 - A_2) U_\alpha F^{\alpha\beta} S_\beta + (A_3 + 1) S_\beta \frac{dU^\beta}{d\tau} = 0$$

If nonelectromagnetic forces are allowed, at least in principle, then $\frac{dU^\beta}{d\tau}$ could be arbitrary

$$\Rightarrow \quad A_1 = A_2$$

$$A_3 = -1$$

$$\frac{dS^\alpha}{d\tau} = A_1 \left\{ F^{\alpha\beta} S_\beta + S_\lambda F^{\lambda\mu} U_\mu U^\alpha \right\} - \frac{1}{c^2} S_\beta \frac{dU^\beta}{d\tau} U_\alpha$$

In rest frame

$$\frac{d\tilde{S}}{d\tau} = A_1 \, F^{\alpha\beta} S_\beta$$

$$\vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}(\beta \cdot s)\vec{\beta} = \vec{s}$$

In rest fram $\frac{d}{d\tau} = \frac{d}{dt}$

compare to

$$\frac{d\vec{s}}{dt'} = \frac{ge}{2mc} \vec{s} \wedge \vec{B}'$$

$$\Rightarrow \quad A_1 = \frac{ge}{2mc}$$

$$\boxed{\frac{dS^\alpha}{d\tau} = \left(\frac{ge}{2mc}\right)\left[F^{\alpha\beta} S_\beta + S_\lambda F^{\lambda\mu} U_\mu U^\alpha\right] - \frac{1}{c^2} S_\beta \frac{dU^\beta}{d\tau} U_\alpha}$$

BMT equation Bargmann, Michel, Telegdi
PRL 2 (1959) 435

Fully relativistic equation to describe how spin of a particle evolves under actions of E+M fields

If $\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$ Lorentz force law

$$\boxed{\frac{dS^\alpha}{d\tau} = \frac{e}{mc}\left[\frac{g}{2} F^{\alpha\beta} S_\beta + \frac{1}{c^2}\left(\frac{g}{2} - 1\right) U^\alpha S_\lambda F^{\lambda\mu} U_\mu\right]}$$

BMT equation

Thomas eq. for precession of spin

Start from BMT eq. and use

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$$

$$\frac{dS^\alpha}{d\tau} = \left(\frac{ge}{2mc}\right)\left[F^{\alpha\beta} S_\beta + \frac{1}{c^2} U^\alpha S_\lambda F^{\lambda\mu} U_\mu\right] - \frac{1}{c^2} U^\alpha S_\lambda \left(\frac{dU^\lambda}{d\tau}\right) \quad (1)$$

Now show

$$S_\lambda \frac{dU^\lambda}{d\tau} = S_0 \frac{dU^0}{d\tau} - \vec{S} \cdot \frac{d}{d\tau}\gamma\vec{v}$$

$$S_0 = \vec{\beta}\cdot\vec{S}$$

$$S_\lambda \frac{dU^\lambda}{d\tau} = \vec{S}\cdot\left[\vec{\beta}\frac{d}{d\tau}\gamma c - \frac{d}{d\tau}(\gamma\vec{v})\right]$$

$$= -\gamma\vec{S}\cdot\frac{d\vec{v}}{d\tau}$$

$$\frac{d\vec{S}}{d\tau} = \vec{F} + \gamma^2\vec{\beta}\left(\vec{S}\cdot\frac{d\vec{\beta}}{d\tau}\right)$$

Here $F^i \equiv \frac{ge}{2mc}\left(F^{i\beta}S_\beta + \frac{1}{c^2}U^i S_\lambda F^{\lambda\mu}U_\mu\right)$

also

$$\frac{dS^0}{d\tau} = F^0 + \gamma^2\left(\vec{S}\cdot\frac{d\vec{\beta}}{d\tau}\right)$$

with

$$F^0 = \left(\frac{ge}{2mc}\right)\left[F^{0\beta}S_\beta + \frac{1}{c^2}U^0 S_\lambda F^{\lambda\mu}U_\mu\right]$$

Rewrite for rest frame spin $\vec{s}$

$$\vec{s} = \vec{S} - \frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{S})\vec{\beta} \qquad 11.158$$

$$= \vec{S} - \left(\frac{\gamma}{\gamma+1}\right) S_0 \vec{\beta}$$

$$\frac{d\vec{s}}{d\tau} = \frac{d\vec{S}}{d\tau} - \frac{\gamma}{\gamma+1}\vec{\beta}\frac{dS_0}{d\tau} - S_0 \frac{d}{d\tau}\left(\frac{\gamma\vec{\beta}}{\gamma+1}\right)$$

gives

$$\frac{d\vec{s}}{d\tau} = \underbrace{\vec{F} - \left(\frac{\gamma\vec{\beta}}{\gamma+1}\right)F_0}_{\vec{F}'} + \frac{\gamma^2}{\gamma+1}\left(\vec{s}\wedge\left(\vec{\beta}\wedge\frac{d\vec{\beta}}{d\tau}\right)\right)$$

$\vec{F}'$ torque in ~~boosted~~ rest frame

$$\gamma\, d\tau = dt$$

$$\frac{d\vec{s}}{dt} = \frac{1}{\gamma}\vec{F}' - \left(\frac{\gamma}{\gamma+1}\right)\left[\left(\beta\wedge\frac{d\vec{\beta}}{d\tau}\right)\wedge\vec{s}\right]$$

Thomas precession freq.

$$\omega_T = \frac{\gamma^2}{\gamma+1}\frac{\vec{a}\wedge\vec{v}}{c^2} \qquad 11.119$$

$$\vec{a} = c\frac{d\vec{\beta}}{dt} = \frac{c}{\gamma}\frac{d\vec{\beta}}{d\tau}$$

$$\frac{d\vec{s}}{dt} = \frac{1}{\gamma}\vec{F}' + \vec{\omega}_T\wedge\vec{s}$$

For motion in E+M fields $\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$

$\Rightarrow$

$$\frac{d\vec{\beta}}{dt} = \frac{e}{\gamma mc}\left[\vec{E} + \vec{\beta}\wedge\vec{B} - \vec{\beta}(\beta\cdot E)\right]$$