Lec. 20 Transformations of E, B

Last time

4 vector current $J^\mu = (c\rho, \vec{J})$

$$\partial_\mu J^\mu = \frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{J} = 0$$

Continuity equation — current conservation

In Lorentz gauge

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$

Define

$$A^\alpha = (\Phi, \vec{A})$$

Maxwell eqs. $\Rightarrow$

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] A^\alpha = \frac{4\pi}{c} J^\alpha$$

Note factor of 4π

We are now using Gaussian units

Before we used SI

With $\partial_\mu A^\mu = 0 = \frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot A$

E, B from A^μ

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Example

$$E_x = \frac{-1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x}$$

$$= -\left(\partial^0 A^1 - \partial^1 A^0\right)$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\left(\partial^2 A^3 - \partial^3 A^2\right)$$

Where $\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla}\right)$

Define field strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

for reference also give $F_{\alpha\beta}$ with two covariant indices

$$F_{\alpha\beta} = g_{\alpha\delta} F^{\delta\gamma} g_{\gamma\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$F_{\alpha\beta}$ is $F^{\alpha\beta}$ with $\vec{E} \to -\vec{E}$.

Can also calculate dual field-strength tensor

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$

Here
$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \alpha=0,\ \beta=1,\ \gamma=2,\ \delta=3 \\ 0 & \text{any indexes are equal} \\ -1 & \text{odd permutation of } 0123 \\ +1 & \text{even permutation} \end{cases}$$

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

From $F^{\alpha\beta}$ ~~replace~~ take $E \to B$ and $B \to -E$

~~Hoping~~ Write Maxwell eqs in covariant form

$$\nabla \cdot E = 4\pi\rho$$

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} J$$

This can be written

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

Homogeneous Maxwell eqs

$$\nabla \cdot B = 0 \,, \qquad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\partial_\alpha \tilde{\mathcal{F}}^{\alpha\beta} = 0$$

Lec. 20 Transformation of E, B

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$U^\alpha = (\gamma c, \gamma \vec{V})$$

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta$$

Lorentz force law

For Macroscopic Maxwell eqs

$$F^{\alpha\beta}(E,B) \Rightarrow G^{\alpha\beta}(D,H)$$

$$G^{\alpha\beta} = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{bmatrix}$$

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$$

Maxwell eqs in macroscopic medium

Polarization $\vec{P}$ and negative of magnetization $-\vec{M}$ transform as a 2nd rank tensor just like $G^{\alpha\beta}$

P, M are macroscopic averages of atomic properties in rest frame of medium.

Transformation of E, B from K to K'

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$$

$$F' = A F \tilde{A}$$

Consider boost along x_1 by $c\beta$ from unprimed to primed frame

$$A = \begin{bmatrix} \gamma & -\gamma\beta_1 & 0 & 0 \\ -\gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & & 0 \\ -\sinh\zeta & \cosh\zeta & & \\ & 0 & 1 & \\ & & & 1 \end{bmatrix}$$

see 11.95 or 11.98

$p = mc \sinh\zeta$ $\qquad E = mc^2 \cosh\zeta$

$\cosh\zeta = \gamma$ $\qquad \sinh\zeta = \gamma\beta$

$$F' = \begin{bmatrix} \gamma & -\gamma\beta_1 & 0 & 0 \\ -\gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta_1 & 0 & 0 \\ -\gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \gamma & -\gamma\beta_1 & 0 & 0 \\ -\gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma\beta_1 E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & -\gamma\beta_1 E_x & -B_z & B_y \\ E_y\gamma - \gamma\beta_1 B_z, & -\gamma\beta_1 E_y + \gamma B_z & 0 & -B_x \\ \gamma E_z + \gamma\beta_1 B_y, & -\gamma\beta_1 E_z - \gamma B_y & B_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{\gamma^2\beta_1 E_x - \gamma^2\beta_1 E_x}^{0}, & -\gamma^2 E_x + \gamma^2\beta_1^2 E_x, & -\gamma E_y + \gamma\beta_1 B_z, & -\gamma E_z - \gamma\beta_1 B_y \\ \gamma E_y - \gamma\beta_1 B_z, & 0, & \gamma\beta_1 E_y - \gamma B_z, & \gamma\beta_1 E_z + \gamma B_y \\ & & 0 & -B_x \\ & & & 0 \end{bmatrix}$$

Compare to

$$F' = \begin{bmatrix} 0, & -E_x', & -E_y', & -E_z', \\ & 0 & -B_z' & B_y' \\ & & 0 & -B_x' \\ & & & 0 \end{bmatrix}$$

$$E_x' = E_x$$
$$E_y' = \gamma(E_y - \beta_1 B_z)$$
$$E_z' = \gamma(E_z + \beta_1 B_y)$$
$$B_x' = B_x$$
$$B_y' = \gamma(B_y + \beta_1 E_z)$$
$$B_z' = \gamma(B_z - \beta_1 E_y)$$

$*$

In general

$$\vec{E}' = \gamma(E + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1} \beta(\vec{\beta} \cdot \vec{B})$$

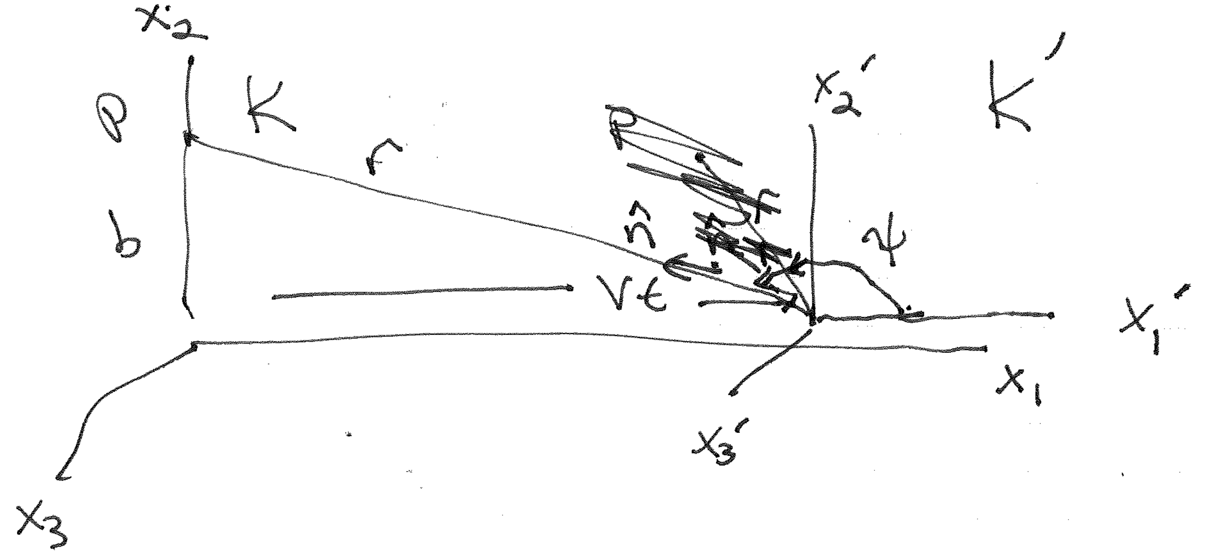
$\vec{E}$ and $\vec{B}$ have no seperate existance.

If $\vec{B} = 0$ in some frame than

$\vec{B} = -\gamma \vec{\beta} \times \vec{E}$

after a boost by $\vec{V}$

Consider a point charge q moving with constant velocity $\vec{v}$



In K' charge is at rest

Let P have coordinates

$$x_1' = -vt'$$

$$x_2' = b$$

$$x_3' = 0$$

$$r' = (b^2 + (vt')^2)^{1/2}$$

distance from origin location of q

In K'

$$E_1' = -\frac{q\,vt'}{r'^3} \qquad E_2' = \frac{qb}{r'^3}, \quad E_3' = 0$$

$$B_1' = B_2' = B_3' = 0$$

Now rewrite in terms of coordinates in frame K.

Location of P in K coordinates

$x_2 = b$
$x_1 = 0$

$t' = \gamma\left(t - \frac{v}{c^2} x_1\right) = \gamma t$ since $x_1 = 0$

$$E_1' = -\frac{q\gamma v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \qquad E_2' = \frac{q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

Electric field at P in frame K' written in terms of coordinates in K

Now boost from K' to K

Use inverse of (*)
take $v \to -v$

$$E_x = E_x'$$
$$E_y = \gamma(E_y' + \beta B_z')$$
$$E_z = \gamma(E_z' - \beta B_y')$$
$$B_x = B_x'$$
$$B_y = \gamma(B_y' - \beta E_z')$$
$$B_z = \gamma(B_z' + \beta E_y')$$

$x = 1$
$y = 2$
$z = 3$

$$E_1 = E_1' = -\frac{q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$E_2 = \gamma E_2' = \frac{q\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \,, \quad B_3' = 0$$

$$E_3 = \gamma E_3' = 0$$

$$B_3 = \gamma \beta E_2' = q\frac{\gamma \beta b}{(b^2+\gamma^2 v^2 t^2)^{3/2}} = \beta E_2$$

$$B_1 = B_2 = 0$$

Nonrelativ. $\Rightarrow$ Biot - Savart expression

$$V/c \ll 1$$

$$B \simeq \frac{q}{c}\frac{\vec{v}\wedge\vec{r}}{r^3}$$

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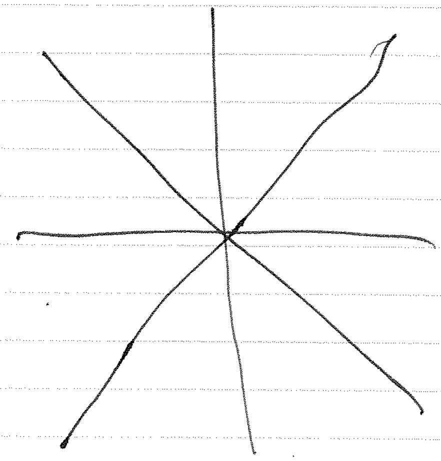
Moving charge is a current and this generates magnetic field.

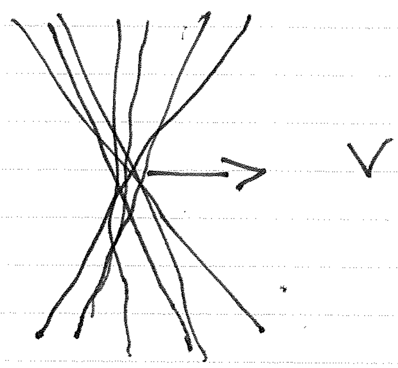
For nonrel. motion

$$\vec{E} \simeq \vec{E}'$$

Little change in electric field for small velocities



Lines of electric force for a particle at rest



Lines of electric ~~field~~ force for a particle moving with $\gamma = 3$

Field lines come from present position of charge

Field gets stronger above and below the particle but

Electric force only acts for a shorter time as localized field quickly passes by

Relativistic motion of heavy ions at RHIC can generate very strong electric and magnetic fields!

A virtual $e^+ e^-$ pair can be spontaneously created out of the vacuum. The e^+ and e^- can be accelerated by the strong $\underline{E}$ field untill they accuire more than $2mc^2$ of energy from the field and become real particles.

This leads to large $e^+ e^-$ pair production.