Lec. 18 Lorentz Transformations

Lab. frame $K \rightarrow$ cm frame $K'$

$\vec{v}_a \rightarrow \vec{b} \rightarrow \vec{v}_c \rightarrow \vec{v}_a' \rightarrow \vec{v}_b' \rightarrow \vec{v}_d \rightarrow \vec{v}_d'$

Frame $K'$ is moving at velocity $\vec{v}$ w.r.t. $K$.

Choose $\vec{v}$ so that $v_b = 0$

$$u_{11} = \frac{u_{11}' + v}{1 + \frac{v \cdot u_{11}'}{c^2}}$$

Addition law for velocities

If $u_{11}' = -V$ then $u_b = \frac{V + V}{1 + \frac{V^2}{c^2}} = 0$

$$u_a = \frac{u_{a1} + v}{1 + \frac{v \cdot u_{a1}}{c^2}} = \frac{2V}{1 + \frac{V^2}{c^2}}$$

Velocity of particle in lab frame.

Last time $\gamma M(u_a) = \left\{ \begin{array}{l} \frac{1 + \beta^2}{1 - \beta^2} \gamma M(0) \\ \frac{\beta}{\beta a} = \gamma M(u_a) \end{array} \right.$
\[
\gamma_a = \left[ \frac{1}{1 - 4\left(\frac{v_a}{c}\right)^2} \right]^{1/2}
\]

\[
\beta = v/c
\]

\[
\gamma = \sqrt{\frac{1}{1 - \beta^2}}
\]

\[
M(u_a) = \gamma_a \cdot m_0 = \gamma_a m
\]

So

\[
\vec{\rho}_a = M(u_a)\vec{u}_a = \gamma_a m \vec{u}_a
\]

Momentum of particle gets extra factor compared to nonrelativistic result.

Last time we wrote in lab frame

\[
U_{cx} = \frac{c\beta \sin \theta}{\sqrt{1 + \beta^2 \cos^2 \theta}}
\]

\[
U_{cz} = c\beta \frac{1 + \cos \theta}{1 - \beta^2 \cos^2 \theta}
\]

\[
U_c = U_{cx}^2 + U_{cz}^2
\]

Work to order \( \theta'^2 \) for small angle

\[
U_{cx}^2 = \frac{c^2 \beta^2 \theta'^2}{(1 + \beta^2)^2}
\]

\[
U_{cz}^2 = c^2 \beta^2 \left[ \frac{1 + 1 - \theta'^2/2}{1 + \beta^2 \left( 1 - \theta'^2/2 \right)^2} \right]^2
\]
\[ u_{c2} = \frac{4 c^2 \beta^2}{\left(1 + \frac{e^2}{2}\right)^2} \left[ 1 - \frac{\beta^2 e^2}{2(1 + \beta^2)^2} \right]^2 \]

\[ u_{c} = \frac{4 c^2 \beta^2}{(1 + \beta^2)^2} \left[ 1 - \frac{\beta^2}{2}\frac{e^2}{1 + \beta^2 e^2} \right] \]

\[ u_{c} = \frac{4 c^2 \beta^2}{(1 + \beta^2)^2} + \frac{c^2 \beta^2 e^2}{(1 + \beta^2)^2} \left[ \frac{1 - \beta^2}{(1 + \beta^2)^2} \right] \]

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\[ u_{ax} = -c \beta \sin \theta \]

\[ u_{dz} = \frac{c \beta (1 - \cos \theta)}{-\beta \cos \theta} \]

\[ u_{dz} = 0 \]

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\[ \gamma^2 = \frac{1}{1 - \beta^2} \]
Conservation of energy

\[ \mathcal{E}(u^a) + \mathcal{E}(o) = \mathcal{E}(u_c) + \mathcal{E}(u_d) \]

Define \( \gamma = \frac{c^2 \beta^2 \sigma^2}{1 - \beta^2} \)

\[ u^a \gamma^2 = u^a - \gamma^2 \left( \frac{1 - \beta^2}{1 + \beta^2} \right) \]

\[ u^a \gamma = \gamma \]

\[ \mathcal{E}(u_a) + \mathcal{E}(o) = \mathcal{E}(u_a^0) - \gamma^2 \left( \frac{\gamma}{\gamma^3} \right) \]

\[ \mathcal{E}(u_a) = \mathcal{E}(o) + \frac{\partial \mathcal{E}}{\partial u^a} |_{u^2} \gamma \]

\[ \mathcal{E}(u_a) + \mathcal{E}(o) = \mathcal{E}(u_a^0) - \frac{\gamma}{\gamma^3} \frac{\partial \mathcal{E}}{\partial u^2} |_{u^2} + \mathcal{E}(o) + \gamma \frac{\partial \mathcal{E}}{\partial u^2} |_{u^2} \]

\[ \Rightarrow \]

\[ \frac{\partial \mathcal{E}}{\partial u^2} |_{u^2} = \gamma^3 \frac{\partial \mathcal{E}}{\partial u^2} |_{0} \]

\[ \frac{\partial \mathcal{E}}{\partial u^2} |_{u^2} = \gamma^3 \frac{m}{2} = \frac{M}{2} \frac{u^2}{(1 - u^2/c^2)^{3/2}} \]

\[ \mathcal{E}(u) = \frac{M c^2}{(1 - u^2/c^2)^{1/2}} + \text{const.} \]
The constant \( C \) can be determined to be zero from experiments as follows:

The \( K^0 \) meson at rest can decay into two \( \pi^0 \) pions, and one can measure the kinetic energy \( E \) of each \( \pi^0 \):

\[ T_{\pi^0} = E_{\pi^0} - E_{K^0} \]

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Measuring \( T_{\pi^0} \) and knowing \( E_{K^0} \) gives:

\[ E_{K^0} = m_{\pi^0} \]

\[ \Rightarrow \quad \text{Const.} = 0 \]

The energy is:

\[ E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Define the four vector \((p_0 = E/c, \vec{p})\):

\[ p^2 = p_0^2 - \vec{p} \cdot \vec{p} = \frac{E^2}{c^2} - \vec{p}^2 \]
\[ p_0^2 - \vec{p} \cdot \vec{p} = \frac{m^2 c^2}{1 - \frac{v^2 c^2}{c^2}} - \frac{m^2 v^2}{1 - \frac{v^2 c^2}{c^2}} \]

\[ p_0^2 - \vec{p}^2 = m^2 c^2 \]

Invariant length of four vector is rest mass of particle

Note \( p_0^2 = p^2 - \vec{p}^2 \) is invariant.

So can evaluate it in any frame. Here, evaluate it in rest frame.

\[ u = 0 \Rightarrow \vec{p} = 0 \]

\[ p_0^2 = E^2 - 0 < m^2 c^2 \]

\[ E = mc^2 \]

Space-Time in Special Relativity

Lorentz transformations are invariant:

\[ s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \]

Assume well-defined transformation that yields new coordinates \( x_0^0, x_1^1, x_2^2, x_3^3 \)

\[ x'^\alpha = x'^\alpha (x_0, x_1, x_2, x_3) \]

\[ \alpha = 0, 1, 2, 3 \]
Tensors of rank k associated with space-time point \( x \) are defined by their transformation properties under \( x \rightarrow x' \):

- **Scalar is unchanged**

- **Vectors** are tensors of rank 1
  - **Contravariant vector** \( A^\alpha \) with four components \( A^0, A^1, A^2, A^3 \)
    
    \[
    A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta 
    \]

  - **Derivative computed from** \( x'^\alpha = x'^\alpha (x^0, ..., x^3) \)
  - **Assume sum over \( \beta \) from 0 to 3**

- **Covariant vector** \( B_\alpha \) defined by

  \[
  B^\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta 
  \]

- A **contravariant tensor of rank 2**

  \[
  F^\alpha{}_{\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'\beta}{\partial x^\delta} F_{\gamma\delta} 
  \]

- A **covariant rank 2 tensor**

  \[
  G^\alpha{}_{\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'_\beta} G_{\gamma\delta} 
  \]

- **Inner product of two vectors** is

  \( B \cdot A = B^\alpha A_\alpha \)

- **Product of contravariant and covariant comp.**
\[ B' \cdot A' = \frac{\partial x^b}{\partial x'^i} \frac{\partial x'^d}{\partial x^y} B_{b} A_{y} = \frac{\partial x^b}{\partial x^y} B_{b} A_{y} = \delta^{y}_{i} B_{b} A_{y} = B_{y} A_{y} = B \cdot A \]

\text{invariant}

Now consider
\[ (ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \]

In general
\[ (ds)^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta \]

\[ g_{00} = 1 \quad g_{11} = g_{22} = g_{33} = -1 \]

Off diagonal elements are
\[ c_{0} = g_{\alpha \beta} / \partial x^\alpha / \partial x^\beta \]

Contravariant \( g^{\alpha\beta} = g_{\alpha\beta} \) for flat space time.

\[ g_{\alpha\beta} \, g^{\gamma\rho} = \delta_{\alpha\beta}^{\gamma\rho} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \]

Assume
\[ A \cdot B = g_{\alpha \beta} A^\alpha B^\beta \]

for \((ds)^2\)

\[ A^\alpha = g_{\alpha \beta} A^\beta \]

and
\[ x^\alpha = g_{\alpha \beta} x^\beta \]

\( g_{\alpha \beta} \) converts contravariant to covariant
\( g^{\alpha \beta} \) converts covariant to contravariant
\[ x^\alpha = g^{\alpha \beta} x_\beta \]

Can raise or lower any index of a tensor with \( g^{\alpha \beta} \) or \( g_{\alpha \beta} \).

\[ F^{\alpha \ldots \alpha} = g^{\alpha \beta} F_{\beta \ldots \beta} \]

If contravariant \( A^\alpha \) has components
\[
(A^0, A^1, A^2, A^3)
\]
its covariant part is \( A_{\alpha} \) has components
\[
(A^0, -A^1, -A^2, -A^3)
\]
\[ A_{\alpha} = (A^0, A^1, A^2, A^3) \quad A^\alpha = (A^0, -A^1, -A^2, -A^3) \]
\[ B \cdot A = B^\alpha A_{\alpha} = \delta^{\alpha}_{\beta} - \delta_{\alpha} B \cdot A \]

Consider partial derivatives
\[ \frac{2}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} \frac{2}{\partial x^\beta} \]

D.f. w.r.t. a contravariant component of a vector transforms like a covariant vector.
\[ \partial x^\alpha = \frac{\partial}{\partial x^\alpha} \quad \frac{\partial}{\partial x^\alpha} = (\frac{\partial}{\partial x^\alpha}, -\nabla) \]
\[ \partial x^\alpha = \frac{\partial}{\partial x^\alpha} = (\frac{\partial}{\partial x^\alpha}, \nabla) \]

4 divergence of a 4-vector.
\[ \partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \mathbf{A} \]

Note plus sign.

4-dimensional Laplacian

\[ \square = \partial^2 + \partial^2 = \frac{\partial^2}{\partial x^0 \partial x^0} - \nabla^2 \]

Matrix representation of Lorentz transformations

Use matrix representation, with components of a contravariant vector forming a column vector

\[ \mathbf{x} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \]

Matrix scalar products

\[ (a, b) \equiv \mathbf{a} \cdot \mathbf{b} \quad \mathbf{a}^T = \text{transpose} \]

Matrix tensor

\[ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Covariant vector

\[ g\mathbf{x} = \begin{pmatrix} x^0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \]
Scalar product of two 4-vectors
\[ a \cdot b = (a, gb) = (ga, b) = \hat{a} g b \]

Find transformation matrix \( A \) that takes \( x \mapsto x' \):
\[ x' = A x \]

Want norm \( x \cdot x \) to be invariant:
\[ x \cdot x = x g x \]
\[ x' \cdot x' = x' g x' = x \cdot x \]
\[ x A^\dagger g A x = x g x \]
\[ \Rightarrow \quad (A^\dagger g A) = g \]

So above can be true for all \( x \)

Properties of \( A \):
\[ \det (A^\dagger g A) = \det g (\det A)^2 = \det g \]
\[ \Rightarrow \quad \det A = \pm 1 \]

Proper Lorentz trans. (boosts, rotations) have \( \det A = 1 \)

Improper Lorentz trans. can have

Example 1: \( \det A = -1 \)

\( A = g \)

But space-time reflection
\[ A = -I \]

\[ x_0 \rightarrow -x_0 \]
\[ x_1 \rightarrow -x_1 \]
\[ x_2 \rightarrow -x_2 \]
\[ x_3 \rightarrow -x_3 \]

is an improper Lorentz trans. with \( \det A = +4 \)

\# of free parameters in \( A \)

\( \tilde{A} \cdot g \cdot A = g \)

equates two \( 4 \times 4 \) matrices

\( \Rightarrow 16 \) equations

However, not all of them are independent because of symmetry under trans positions.

\[ 16 - (1 + 2 + 3) = 10 \]

independent equations. Thus \( 4 \times 4 \) matrix has 16 numbers less 10 constraints

\( \Rightarrow \) Need 6 parameters to specify \( A \)

example 3 comp. of boost velocity and 3 angles of rotation.

end 3/31/11