Read 10.1, 10.2 and 10.5 3/3/11

Lec. 14 Perturbation Theory of Scattering and the Blue Sky

Consider a medium with small fluctuations in ϵ and μ. Calculate ~~scattered~~ scattering from these fluctuations

Start from

$$\nabla \cdot B = 0 \qquad \nabla \times E = -\partial B/\partial t$$
$$\nabla \cdot D = 0 \qquad \nabla \times H = \frac{\partial D}{\partial t}$$

Write

$$D = \epsilon E \qquad \epsilon = \epsilon_0 + \delta\epsilon$$
$$B = \mu H \qquad \mu = \mu_0 + \delta\mu$$

Work to lowest order in $\delta\epsilon$, $\delta\mu$

Note ϵ_0, μ_0 are values for average medium. Not necessarily free space values!

$$\cancel{\nabla^2 D - \mu_0 \epsilon_0 \frac{\partial^2 D}{\partial t^2}}$$

$$\nabla \times \nabla \times E = -\frac{\partial}{\partial t}(\nabla \times B)$$

Multiply by ϵ_0

$$\nabla \times \nabla \times \epsilon_0 E = -\epsilon_0 \frac{\partial}{\partial t}(\nabla \times B)$$

add and subtract $\nabla \times \nabla \times D$ and $\epsilon_0 \frac{\partial}{\partial t} \nabla \times \mu_0 H$

$$\nabla \times \nabla \times D = \nabla \times \nabla \times (D - \epsilon_0 E) - \epsilon_0 \frac{\partial}{\partial t} \nabla \times (B - \mu_0 H)$$
$$- \epsilon_0 \mu_0 \frac{\partial}{\partial t}(\nabla \times H)$$

$$-\nabla \times \nabla \times D - \epsilon_0 \mu_0 \frac{\partial^2 D}{\partial t^2} = -\nabla \times \nabla \times (D - \epsilon_0 E)$$
$$+ \epsilon_0 \frac{\partial}{\partial t} \nabla \times (B - \mu_0 H)$$

$$\nabla \times \nabla \times D = -\nabla^2 D + \nabla(\nabla \cdot D) = -\nabla^2 D$$

So

$$\left[\nabla^2 - \epsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right] D = -\nabla\times\nabla\times(D-\epsilon_0 E) + \epsilon_0 \frac{\partial}{\partial t}\nabla\times(B-\mu_0 H)$$

Consider time indep. fluctuations $\delta\epsilon = \delta\epsilon(\vec{x})$
and time dep. $e^{-i\omega t}$ $k^2 = \mu_0\epsilon_0\omega^2$

$$\left[\nabla^2 + k^2\right] D = -\nabla\times\nabla\times(D-\epsilon_0 E) - i\omega\epsilon_0 \nabla\times(B-\mu_0 H)$$

Consider an ~~incoming~~ initial plane wave that satisfies

$$\left[\nabla^2 + k^2\right] D^{(0)} = 0$$

and solve with green func.

$$\boxed{D(\vec{x}) = D^{(0)} + \frac{1}{4\pi}\int d^3x' \, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \left[\nabla'\times\nabla'\times(D-\epsilon_0 E) + i\epsilon_0\omega \nabla\times(B-\mu_0 H)\right]}$$

(1)

Consider $\delta\epsilon, \delta\mu$ confined to a finite region of space.

$$D \to D^{(0)} + A_{sc}\frac{e^{ikr}}{r}$$

for large r

$$A_{sc} = \frac{1}{4\pi}\int d^3x' \, e^{-ik\,n\cdot x'} \left[\nabla'\wedge\nabla'\wedge(D-\epsilon_0 E) + i\omega\epsilon_0\nabla'\wedge(B-\mu_0 H)\right]$$

integrate by parts $\nabla' e^{-ik\,n\cdot x'} = -ik\vec{n}\, e^{-ik\,n\cdot x'}$

$$A_{sc} = \frac{k^2}{4\pi}\int d^3x' \, e^{-ik\,n\cdot x'} \left\{\left[n\wedge(D-\epsilon_0 E)\right]\wedge n - \frac{\epsilon_0\omega}{k}\, n\wedge(B-\mu_0 H)\right\}$$

(2)

This ~~sc~~ scattering amplitude gives

$$\frac{d\sigma}{d\Omega} = \frac{|\epsilon^* \cdot A_{sc}|^2}{|\vec{D}^{(0)}|^2} \qquad (3)$$

Equations 1, 2, 3 provide a formal solution to scattering problem. Note need full solution for D, H to evaluate RHS of (2)

Born Approximation

$$D(x) = [\epsilon_0 + \delta\epsilon(x)]\, E(x)$$

$$B = [\mu_0 + \delta\mu(x)]\, H(x)$$

$$D - \epsilon_0 E \approx \frac{\delta\epsilon(x)}{\epsilon_0} D^{(0)}(x)$$

$$B - \mu_0 H \approx \delta\mu(x)\, H^{(0)}(x)$$

plane wave

$$D^{(0)}(x) = \vec{\epsilon}_0\, D_0\, e^{ik n_0 \cdot x}$$

note polarization vector $\vec{\epsilon}_0$

$$B^{(0)} = \sqrt{\frac{\mu_0}{\epsilon_0}}\; n_0 \wedge D^{(0)}$$

Put these into (2)

$$\frac{\epsilon^* \cdot A^{(1)}_{sc}}{D_0} = \frac{k^2}{4\pi} \int d^3x\, e^{i\vec{q}\cdot\vec{x}} \Big[\epsilon^* \cdot \epsilon_0\, \delta\epsilon(x)/\epsilon_0 + (n \wedge \epsilon^*)(n_0 \wedge \epsilon_0)\, \frac{\delta\mu(x)}{\mu_0} \Big]$$

with

$$\vec{q} = k(\vec{n}_0 - \vec{n})$$

Blue Sky: Elementary Argument

Lord Rayleigh first described quantitatively the scattering of light by gases, and explained sunsets and the blue sky

Magnetic moments are small $\delta\mu \simeq 0$

$$\delta \epsilon(x) = \epsilon_0 \sum_j \gamma_{mol}\, \delta(x - x_j)$$

For individual molecules ~~where~~ that have electric dipole moments

$$\vec{p}_j = \epsilon_0 \gamma_{mol} E(\vec{x}_j)$$

Plug this into

$$\frac{\epsilon^* \cdot A_{sc}}{D_0} = \frac{\gamma_{mol} k^2}{4\pi} \int d^3x \, e^{i q\cdot x} \, \vec{\epsilon}^* \cdot \vec{\epsilon}_0 \sum_j \delta(x - x_j)$$

$$= \frac{\gamma_{mol} k^2}{4\pi} \sum_j e^{i q \cdot x_j} \, \vec{\epsilon}^* \cdot \vec{\epsilon}_0$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{k^4 \gamma_{mol}^2}{16\pi^2} \left| \vec{\epsilon}^* \cdot \vec{\epsilon}_0 \right|^2 F(q)}$$

where $F(q)$ = Structure factor

$$= \Big|\sum_j e^{i q \cdot x_j}\Big|^2 = \sum_{i,j=1}^{N} e^{i \vec{q} \cdot (x_i - x_j)}$$