Lec. 13 Review
Midterm 3/1/11

2/24/11
Collection of Scatters

Let \( x_j \) be the position of \( j \)th scatter. Its moments will have a phase \( e^{ik\cdot n \cdot x_j} \) and if the obserer is far from the scatterers than their fields will have a factor \( e^{-ik\cdot n \cdot x_j} \).

\[
E_{sc} = \frac{1}{4\pi\epsilon_0} \frac{k^2}{r} \frac{e^{ik\cdot r}}{r} \left[ \sum_{\mathbf{m}n} n_{\mathbf{m}n} - n_{\mathbf{m}0} \right] \\
\mathbf{r} = |\mathbf{x} - \mathbf{A} \cdot x_j|
\]

If the center of distribution of scatterers is at \( \mathbf{x}_0 \)

\[
\frac{d\sigma}{d\mathbf{q}} = \frac{4}{(4\pi\epsilon_0)^2} \sum_j \left[ \mathbf{E}^* \cdot \mathbf{p}_j + (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{m}_j \right] e^{i\mathbf{q} \cdot x_j} \frac{1}{4\pi\epsilon_0} \frac{k^2}{|\mathbf{r}|} \frac{e^{ik\cdot r}}{r} \\
\mathbf{q} = k \mathbf{n}_0 - k \mathbf{\hat{n}}
\]

If all scatterers are identical then

\[
\frac{d\sigma}{d\mathbf{q}} = \frac{d\sigma}{d\mathbf{q}} \bigg|_{\text{F(q)}}
\]

\( F(q) \approx \left| \sum_j e^{i\mathbf{q} \cdot x_j} \right|^2 = \sum_j \sum_{j'} e^{i\mathbf{q} \cdot (x_j - x_{j'})} \)

If positions of scatterers random then

\[
\sum_j e^{i\mathbf{q} \cdot x_j} \approx 0 \text{ if } i \neq j
\]
\[ F(q) \rightarrow N \]

In coherent scattering from \( N \) objects
\[ \frac{d\sigma}{d\Omega} = N \frac{d\sigma}{d\Omega} ! \]

In short wave length limit of scattering from a regular array of scatterers, \( F(q) \) has Bragg peaks when
\[ q \cdot a = 0, \ \frac{a}{2}, \ldots \]

Consider a simple cubic array of scatterers with lattice spacing \( a \) and lattice sites in each
\[ F(q) = N^2 \left\{ \frac{3 \sin^2 (Nq_1 a)}{2} \frac{3 \sin^2 (Nq_2 a)}{2} \frac{3 \sin^2 (Nq_3 a)}{2} \right\} \]

\[ \frac{N^2 \sin^2 q_1 a}{N^2 \sin^2 q_2 a} \frac{N^2 \sin^2 q_3 a}{N^2 \sin^2 q_4 a} \]

In long wave length limit
\[ F \sim N^2 \left( \sin \left( \frac{4\pi x_i}{2} \right) \right)^2 \]

\[ x_i = q_i a N_i \]

This can be big only in forward direction
\[ \theta \leq \frac{x}{L} \]

Where \( L \sim N a \) is the size of the system.
Coherent neutrino scattering

In 1976's weak neutral currents were discovered where a $\nu$ can exchange a Z boson with a nucleon.

\[ \nu \rightarrow \nu \rightarrow p \]

The cross section is small

\[ \frac{d\sigma}{dQ^2} \propto \frac{G^2 F^2}{Q^2} \]

Where $G_F$ is the Fermi constant of beta decay.

Question: Can a $\nu$ scatter coherently from a crystal?

\[ \frac{d\sigma}{dQ^2} = N^2 \cdot F(q) \cdot \frac{d\sigma}{dQ^2} \]

If $N = 10^{23}$ this could be a big cross section.

Yes but (1) regular array of atoms (2) very small scattering angle

If the $\nu$, scattered at a very small angle, how do you tell???
Review

Some definition/short answer questions but probably less than in P506
Short simplified homework problems
Core material so far in P507
Note lectures 1-10 on course website
http://cecelia.physics.indiana.edu/p507
lec 11, 12 will be posted Friday.

Chap. 7 Plane Waves
and Wave propagation

Maxwell eqs in absence of sources
\[ \nabla \cdot B = 0 \quad \nabla \times E + \partial B/\partial t = 0 \]
\[ \nabla \cdot D = 0 \quad \nabla \times H - \partial D/\partial t = 0 \]
Assume \( e^{-iwt} \) time dep. and
\[ D = \varepsilon E \quad B = \mu H \]
\[ \nabla \times E = -i\omega B \quad \nabla \times B = i\omega \mu E = 0 \]

\[ \nabla^2 + \omega^2 \mu \varepsilon] E = 0 \]
\[ \nabla^2 + \omega^2 \mu \varepsilon B = 0 \]
\[ k = \sqrt{\mu \varepsilon \omega} \quad \text{wave vector} \]
\[ \nu = \frac{\omega}{k} \quad \text{phase velocity} \]
\[ V = \sqrt{\mu \varepsilon}, \quad n = \sqrt{\mu \varepsilon / \varepsilon_0} \] index of refraction

Plane waves with \( \mathbf{E} \perp \mathbf{B} \perp \hat{n} \)

Polarization and Stokes parameters linear, circular, elliptical

Reflection / Refraction at plane interface between dielectrics Boundary conditions Normal comp. of \( \mathbf{D}, \mathbf{B} \) are cont.

Tangential comp. of \( \mathbf{E}, \mathbf{H} \) are cont.

Snell's law \( \frac{\sin i}{\sin o} = \frac{n'}{n} \)

Dispersion Freq. dep. of \( n \) and

Model for dielectric \( \varepsilon(\omega) \) [Section 7.5]

Collection of harmonic osc. with freq. \( \omega \); damping \( \gamma_i \) and strength \( f_i \):

\[ \varepsilon(\omega) = 1 + \sum \frac{N \varepsilon^3 \varepsilon}{\varepsilon_0 \omega m} j \left( \frac{\omega^2 - \omega^2 - i \omega \gamma_i}{\omega^2} \right) \]

\( N \) molecules per unit volume, Sum rule \( \sum f_i = \varepsilon \)
Away from a resonance $\omega \neq \omega_j$

Normal dispersion

$\varepsilon(\omega)$ increases with increasing $\omega$

Anomalous dispersion

$\text{Re } \varepsilon(\omega)$ has rapid $\omega$ dep.

Near a resonance $\omega = \omega_j$ where $\text{Im } \varepsilon(\omega)$ is large

Fig 7.8

$\text{Re } \varepsilon$

$\text{Im } \varepsilon$

Low freq. dep. and electrical cond.

High freq. dep. and plasma freq.

For $\omega \gg \text{all } \omega_j$
\[ E = 1 \text{ } \text{ } \sqrt{\frac{\omega_p^2}{\omega^2}} \]

**Plasma freq.** \( \omega_p \)

\[ \omega_p^2 = \frac{N Z e^2}{\varepsilon_0 m} \]

Depends on charge density and mass of charged particles (electrons).

Index of refraction and absorption coeff. of liquid and water

Review Fig 7.9

Dependence of absorption on frequency

Absorb. is small only for narrow range of visible freq. at higher and lower freq.

\[ n \approx 1.34 \text{ } \text{ } \text{low freq.} \]

**Magnetohydrodynamics**

In a conducting fluid or ionized gas, collisions are so rapid that Ohms law will hold

\[ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \]

Where Ohms law is generalized to include bulk motion \( V \) of fluid.
Hydrodynamic eq.

cont. eq. \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \)

Newton's 2nd law

\( \frac{d}{dt} \mathbf{p} = F \)

\( \rho \frac{d\mathbf{v}}{dt} + \mathbf{p} \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \frac{1}{\rho} \mathbf{J} \times \mathbf{B} \)

Magnetic force density with \( \mathbf{J} = 0 \) inside conductor

Alfven waves can propagate along

\( \mathbf{B} \) with velocity

\[ \mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{\mu \rho_0}} \]

\( \rho_0 = \) equilibrium density

1.8 Superposition of waves and group velocity

\[ u(x, t) = \int \overline{A(k)} e^{i(kx - \omega(k)t)} dk \]

\[ v_g = \frac{\omega}{k} = \frac{c}{n(\omega) + \frac{\omega}{\overline{\epsilon}} \frac{\partial n}{\partial \omega}} \]
Spreading of a pulse as it moves in a dispersive medium.

Kraemer - Kronig Relations

Causality

\( \frac{E(\omega)}{E_0} \) is analytic in the upper half plane.

\[ D(x,t) = E(\omega) \, E(\ast, \omega) \]

\( \mathcal{F} \), T.

\( D(x,t) \) can only depend on \( E(x,t') \) for, \( t' \leq t \)

\[ E(-\omega) / E_0 = E^*(\omega) / E_0 \]

For real \( \omega \)

\[ \text{Im} E(-\omega) = -\text{Im} E(\omega) \]

\[ \text{Re} E(-\omega) = \text{Re} E(\omega) \]

Dispersion Relation

\[ \frac{\text{Re} E(\omega)}{E_0} = 1 + 2 \pi \sum_{\omega} \frac{\text{Im} E(\omega')/E_0}{\omega^2 - \omega'^2} \]

\[ \frac{\text{Im} E(\omega)}{E_0} = -2 \pi \sum_{\omega} \frac{\text{Re} E(\omega')/E_0}{\omega^2 - \omega'^2} \]

Similar relations are many areas of physics.
Chapter 9

Radiating Systems

Use retarded Green's Func.

\[ \lim_{kr \to \infty} A(x) = \frac{-\mu_0}{4\pi} \int_{\mathbb{R}^3} J(x') e^{-ikr} e^{i\Phi} d^3x' \]

Electric dipole fields

\[ H = \frac{e}{2\pi} \frac{n \times p}{r} e^{i\Phi} \]

\[ E = \frac{Z_0}{n} H \wedge n \]

\[ \mathcal{P} = \int \mathbf{p}(x') d^3x' \] electric dipole moment

Similar formulae for magnetic dipole

\[ \frac{d\mathcal{P}}{dt} \text{ radiated} \]

\[ \left[ \frac{\partial}{\partial t} \right] \mathcal{E} = \frac{i}{2} \text{Re} \left[ \mathbf{E} \cdot \mathbf{H}^* \right] \]

Spherical Wave solutions to Scalar Wave

\[ \left( \nabla^2 + k^2 \right) \psi(x, \omega) = 0 \quad \text{if} \quad \mathbf{k} = \omega / c \]

\[ \psi = \sum_{lm} f_l^m \ Y_l^m \]

\[ f_l^m \alpha \ j_l^m(kr) \quad \text{or} \quad p_l^m(kr) \]

Spherical Bessel Func. \[ j_l^m = \frac{1}{2} i^l \frac{d^l}{dr^l} \left[ r^l \right] \]
\[
\frac{e^{ik|x-x'|}}{4\pi |x-x'|} = i\frac{k}{2} \int \frac{1}{2} \left( \frac{\partial}{\partial n} \right) \nabla e^{ikr} \mathrm{d}S
\]

\[
L = \frac{1}{i} \nabla
\]

\[
X_{lm} = \frac{L Y_{lm}}{\sqrt{8\pi (l+1)}}
\]

\[
H = \sum \alpha(lm) \frac{1}{k} X_{lm} - i \sum \alpha(lm) \nabla g(x) \frac{1}{k} X_{lm}
\]

\(E\) related to curl \(H\)

Expansion of general solution of Maxwell's Eq in source free region

In \(r \to \infty\) limit \(\alpha(lm) \to \frac{l+1}{k}\) \(h_{lm} = \frac{l+1}{k} e^{ikr}\)

\[
H \to e^{ikr-i\omega t} \sum \alpha(lm) \frac{1}{k} X_{lm} + \alpha(lm) \nabla \times X_{lm} \nabla
\]

\(E \to 0\) \(\nabla = 0\)

Sources of moments multipole radiation multipole

\[
a_E = \frac{k^2}{i\omega} \sum_{lm} \frac{1}{2} \epsilon_{ij} \frac{\partial}{\partial x_j} \phi_{lm}(x) e^{ikr} \int \frac{e^{ikr}}{r} \mathrm{d}S
\]

\[
-ik \nabla \cdot (\gamma_{0m}) j_x \nabla x
\]

Similar exp. for \(a_m\)

Power radiated angular dist. in terms of \(a_E, a_m\)
Scattering and Diffraction

Incident fields induce $\mathbf{p}$ and $\mathbf{m}$ and they then radiate in scattered directions...