2/22/11

Lec. 12 Scattering

Last time Cente fed antenna

Exact result

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right|^2$$

$$J = \hat{r}\, \frac{I(r)}{2\pi r^2} \left\{ \delta(\cos\theta - 1) - \delta(\cos\theta + 1) \right\} \qquad r < d/2$$

$$\rho = \frac{1}{i\omega} \frac{\partial I}{\partial r} \left(\frac{\delta(\cos\theta - 1) - \delta(\cos\theta + 1)}{2\pi r^2} \right)$$

$$a_m = \frac{k^2}{i\sqrt{l(l+1)}} \int Y^*_{lm}\, \nabla \cdot (r \wedge J)\, j_l(kr)\, d^3x \equiv 0$$

$$a_E(lm) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y^*_{lm} \left\{ c\rho \frac{\partial}{\partial r}(r j_l) + ik\, \vec{r} \cdot \vec{J}\, j_l(kr) \right\} d^3r$$

$$a_E = \frac{k^2}{2\pi\sqrt{l(l+1)}} \int_0^{d/2} dr \left[kr\, j_l(kr)\, I(r) - \frac{1}{k}\frac{\partial I}{\partial r}\frac{\partial}{\partial r}(r j_l) \right]$$

$$\int d\Omega\, Y^*_{lm} \left[\delta(\cos\theta - 1) - \delta(\cos\theta + 1) \right]$$

$\boxed{\text{end } 2/17/11}$

$$\int d\Omega\, Y^*_{lm} (\delta - \delta) = 2\pi\, \delta_{m,0} \left[Y_{l0}(0) - Y_{l0}(\pi) \right]$$

only $m=0$ multipoles. Note only nonzero for l odd.

$$\int \Omega\, Y^*_{lm}(\delta - \delta) = \sqrt{4\pi(2l+1)}\ \delta_{m,0} \text{ and } l \text{ odd}$$

$$a_E(l,0) = \frac{k}{2\pi}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}} \int_0^{d/2} dr \left\{ -\frac{d}{dr}\left[r j_l \frac{\partial I}{\partial r} \right] + r\, j_l(kr)\left(\frac{\partial^2 I}{\partial r^2} + k^2 I \right) \right\}$$

Now need $I(r)$

Let us assume $I(z) = I \sin\left(\frac{kd}{2} - k|z| \right)$

$$= -\frac{k}{2\pi}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}}\ \frac{d}{2}\, j_l\!\left(\frac{kd}{2}\right) I \cos\left(\frac{kd}{2} - k|z| \right)(-k) \Big|^{z=d/2}$$

$$\boxed{a_E(l,0) = \frac{I}{\pi d}\left(\frac{kd}{2}\right)^2 \sqrt{\frac{4\pi(2l+1)}{l(l+1)}}\ j_l\!\left(\frac{kd}{2}\right)} \quad l \text{ odd}$$

Example let $kd = 2\pi$ not $\ll 1$

$$a_E(1,0) = \frac{I}{d}\,\pi \sqrt{\frac{4\pi \cdot 3}{2}}\ j_1(\pi)$$

$$j(\pi) = \frac{\sin x}{x^2} - \frac{\cos x}{x} = \frac{1}{\pi}$$

$$\Rightarrow \quad a_E(1,0) = \frac{I}{d}\sqrt{6\pi}$$

$$a_{\cancel{E}}(3,0) = \pi\frac{I}{d}\sqrt{\frac{4\pi \cdot 7}{3 \cdot 4}}\; j_3(\pi)$$

$$j_3(\pi) = \left(\frac{15}{x^4} - \frac{6}{\cancel{\pi}^2}\right)\sin x - \left(\frac{15}{x^3} - \frac{1}{x}\right)\cos x$$

$$= \frac{15}{\pi^3} - \frac{1}{\pi}$$

$$a_E(3,0) = \frac{I}{d}\left(\frac{15}{\pi^2} - 1\right)\sqrt{\frac{7\pi}{3}}$$

$$\frac{a_E(3,0)}{a_E(1,0)} = \frac{\sqrt{\frac{7\pi}{3}}\left(\frac{15}{\pi^2} - 1\right)}{\sqrt{6\pi}} = \sqrt{\frac{7}{2}}\;\frac{1}{3}\left(\frac{15}{\pi^2} - 1\right) = 0.3242$$

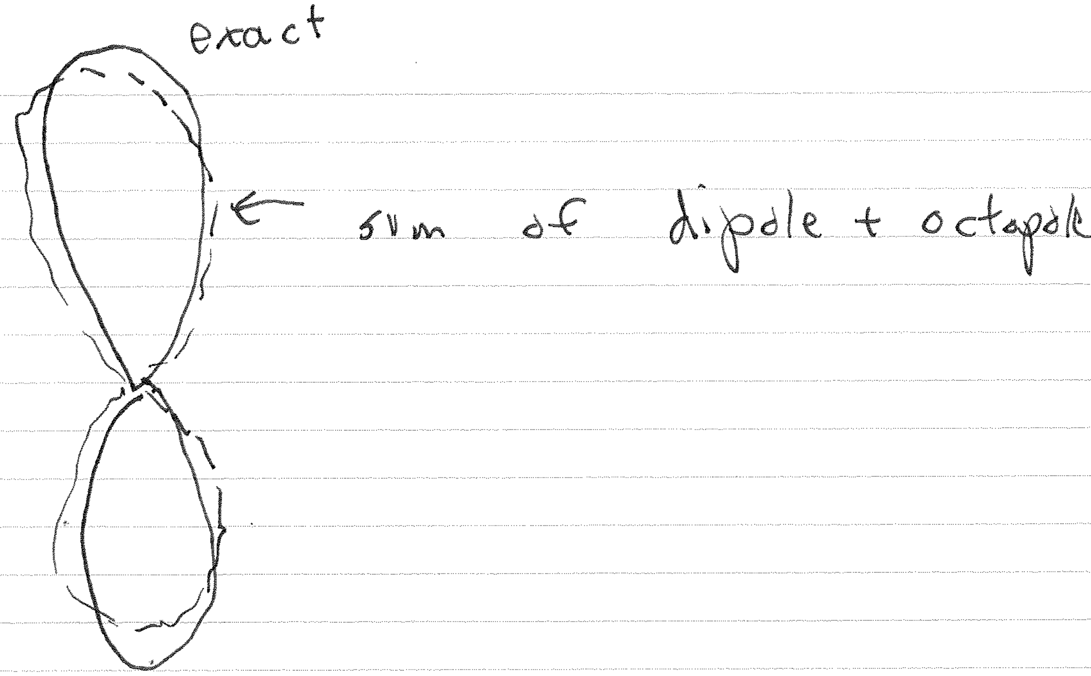
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm} (-i)^{l+1}\left[a_E(l,m) X_{lm} \, a\, n + a_m X_{ln}\right]\right|^2$$ 9.150

$$|LY_{10}|^2 = \frac{3}{4\pi}\sin^2\theta \qquad (LY^*_{10})\, LY_{30} = \frac{3\sqrt{21}}{8\pi}\sin^2\theta\,(5\cos^2\theta - 1)$$

$$|LY_{30}|^2 = \frac{63}{16\pi}\sin^2\theta\,(5\cos^2\theta - 1)^2$$

$$\frac{dP}{d\Omega} = \frac{Z_0|a_E(1,0)|^2}{4k^2}\left| LY_{10} - \frac{a_E(3,0)}{\sqrt{6}\, a_E(1,0)} LY(3,0)\right|^2$$

$P(\theta)$



$$\left.\frac{dP}{d\Omega}\right|_{\text{exact}} = \frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos(\pi\cos\theta)+1}{\sin\theta}\right|^2 = \frac{Z_0 I^2}{8\pi^2}\left[\frac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right]$$

Expansion works well even when $kd \sim 1$

Chapter 10

Scattering and Diffraction

Consider scattering at long wavelengths
→ dominated by dipole ~~moment~~ moments

$$E_{inc} = \epsilon_0 E_0 e^{ik n_0 \cdot x}$$

$$H_{inc} = n_0 \wedge E_{inc} / Z_0$$

gives rise to electric dipole moment $\vec{p}$
and magnetic dipole moment $\vec{m}$
Note $e^{-i\omega t}$ understood.

$$E_{sc} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r}\left[(n \wedge p)\wedge n - n \wedge \frac{m}{c}\right]$$

in large r limit 9.19, 9.36

$H_{sc} = n \wedge E_{sc} / Z_0$

n in direction of observation

$$\frac{d\sigma}{d\Omega}(n, \epsilon ; n_0, \epsilon_0) = \frac{r^2 \frac{1}{2Z_0} |\epsilon^* \cdot E_{sc}|^2}{\frac{1}{2Z_0} |\epsilon_0^* \cdot E_{inc}|^2}$$

Power radiated into solid angle $d\Omega$ ~~per unit area~~ in direction n divided by incident flux (power per unit area) is dif. cross section.

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} \left| \epsilon^* \cdot p + (n \wedge \epsilon^*) \cdot \frac{m}{c} \right|^2$$

$\vec{p}, \vec{m}$ depend on E_0, $\vec{n}_0$ and $\vec{\epsilon}_0$

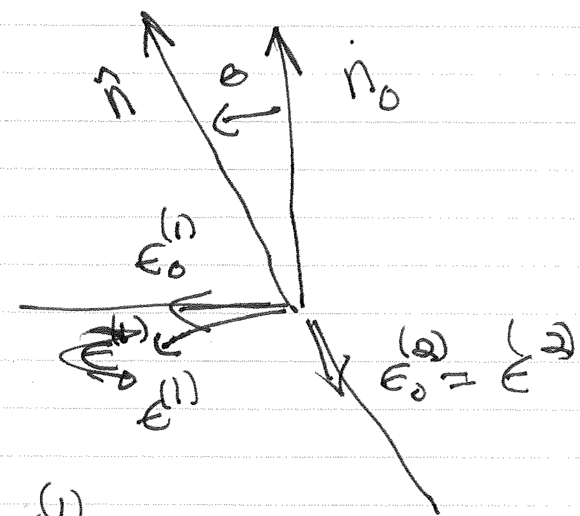
In long wavelength limit expect ω^4 dependence on frequency.

Scattering by a small Dielectric sphere

$$\vec{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) a^3 \vec{E}_{inc} \qquad (4.56)$$

Induced electric dipole moment of ~~die~~ dielectric sphere, $\vec{m} = 0$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 |\epsilon^* \cdot \epsilon_0|^2$$



$$\epsilon^{(1)} \cdot \epsilon_0^{(1)} = \cos\theta$$
$$\epsilon^{(1)} \cdot \epsilon_0^{(2)} = 0$$
$$\epsilon^{(2)} \cdot \epsilon_0^{(2)} = 1$$
$$\epsilon^{(2)} \cdot \epsilon_0^{(1)} = 0$$

For an unpolarized initial beam

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \frac{1}{2}\left[|\epsilon^* \cdot \epsilon_0^1|^2 + |\epsilon^* \cdot \epsilon_0^2|^2\right]$$

average over initial polarization.

Calculate polarization in the scattering plane $\epsilon = \epsilon^{(1)}$

$$\frac{d\sigma}{d\Omega}_{\parallel} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \frac{\cos^2\theta}{2}$$

$$\frac{d\sigma}{d\Omega}_{\perp} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \frac{1}{2}$$

$$P(\theta) = \frac{\frac{d\sigma}{d\Omega_\perp} - \frac{d\sigma}{d\Omega_\parallel}}{\frac{d\sigma}{d\Omega_\perp} + \frac{d\sigma}{d\Omega_\parallel}} = \frac{\sin^2\theta}{1 + \cos^2\theta}$$

scattered radiation is polarized.

Total dif. cross sec. is summed over ~~initial~~ final polarizations

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \left(\frac{1+\cos^2\theta}{2}\right)$$

Total cross sec. is

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi \left(\frac{1 + \langle \cos^2\theta \rangle}{2}\right) k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2$$

$$\langle \cos^2\theta \rangle = \int_{-1}^{1} d\cos\theta \, \cos^2\theta \Big/ \int_{-1}^{1} d\cos\theta$$
$$= \frac{1}{3}$$

$$\sigma = \frac{8\pi}{3} k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2$$

<u>Scattering by conducting sphere</u>
(assume small, high conductivity)

Electric dipole moment in section 2.5

$$p = 4\pi \epsilon_0 a^3 E_{inc}$$

(No electric field inside conductor)

The conducting sphere also has a magnetic dipole moment (different from a dielectric sphere)

Boundary condition normal component of B vanishes at $r = a$ (for a perfect conductor). Simple direct calculation gives

$$m = -2\pi a^3 H_{inc}$$

For a linearly polarized incident wave

$\vec{m}$ $\vec{p}$ at right angles to each other and to the incident direction.

$$\frac{d\sigma}{d\Omega}(n\,\epsilon;\,n_0,\epsilon_0) = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\epsilon^*\cdot p + n\wedge\epsilon^*\cdot\frac{m}{c}\right|^2$$

$$\vec{E}_{inc} = \epsilon_0 E_0\, e^{ikn_0\cdot x} \qquad \vec{p} = (4\pi\epsilon_0)\,a^3\,\vec{\epsilon}_0\,E_0 e^{ikn_0\cdot x}$$

$$H_{inc} = (n_0\times\epsilon_0)\frac{E_0}{Z_0}\,e^{ikn_0\cdot x} \qquad \vec{m} = -2\pi a^3(n_0\wedge\epsilon_0)\frac{E_0}{Z_0}e^{ikn_0\cdot x}$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6\left|\epsilon^*\cdot\epsilon_0 - \frac{1}{2\epsilon_0 Z_0 c}(n\wedge\epsilon^*)\cdot(n_0\wedge\epsilon_0)\right|^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6\left|\epsilon^*\cdot\epsilon_0 - \frac{1}{2}(n\wedge\epsilon^*)\cdot(n_0\wedge\epsilon_0)\right|^2}$$

$$\epsilon^{*1}\cdot\epsilon_0^1 = \cos\theta \qquad (n_0\wedge\epsilon_0^1)\cdot n\wedge\epsilon^1 = 1$$

$$\epsilon^{*2}\cdot\epsilon_0^2 = 1 \qquad (n_0\wedge\epsilon_0^2)\cdot(n\wedge\epsilon^2) = \cos\theta$$

$$\epsilon^{*1}\cdot\epsilon_0^2 = 0$$

unpolarized incident beam

$$\frac{d\sigma}{d\Omega}_{\parallel} = \frac{k^4a^6}{2}\left\{\left|\cos\theta - \frac{1}{2}\right|^2\right\} \cancel{+\left\{\left[1-\frac{1}{2}\cos\theta\right]^2\right\}}$$

$$\frac{d\sigma}{d\Omega}_{\perp} = \frac{k^4a^6}{2}\left\{1-\frac{1}{2}\cos\theta\right\}^2$$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\parallel} + \frac{d\sigma}{d\Omega}_{\perp} = k^4a^6\left[\cos^2\theta - \cos\theta + \frac{1}{4} + 1 - \cos\theta + \frac{1}{4}\cos^2\theta\right]\frac{1}{2}$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6\left[\frac{5}{8}(\cos^2\theta+1) - \cos\theta\right]$$

Cross section is peaked at backward scattering angles $\theta\to 180^\circ$ because of electric dipole and magnetic dipole interference.