2/17/11

Lec. 11 Calculating Multipole Moments

Midterm 3/1

Can project multipole coef. a_E, a_M in expansion

$$H = \sum_{lm} [a_E f_l \vec{X}_{lm} - \frac{i}{k} a_M \nabla_\wedge g_l \vec{X}_{lm}]$$

$$E = \frac{i}{k} Z_0 \nabla_\wedge H$$

from

$$a_M h_l^1(kr) = \frac{k}{\sqrt{l(l+1)}} \int d\Omega\, Y^*_{lm} \vec{r}\cdot\vec{H}$$

$$Z_0 a_E h_l^1(kr) = -\frac{k}{\sqrt{l(l+1)}} \int d\Omega\, Y^*_{lm} \vec{r}\cdot\vec{E}$$

$$(\nabla^2 + k^2)\, r\cdot H' = -i L\cdot(J + \nabla_\wedge \mathcal{M})$$

with $H' = B/\mu_0$ localized sources $\rho, J, \mathcal{M}$

$$r\cdot H'(x) = \frac{i}{4\pi}\int \frac{e^{ikR}}{R} L'\cdot(J + \nabla_\wedge \mathcal{M})\, d^3x$$

$$\frac{e^{ikR}}{4\pi R} = ik \sum_{lm} j_l(kr_<)\, h_l^1(kr_>)\, Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)$$

$$\boxed{\begin{aligned} a_E(lm) &= \frac{ik^3}{\sqrt{l(l+1)}} \int d^3r'\, j_l(kr')\, Y^*_{lm}(\theta'\phi')\, L'\cdot[\mathcal{M} + \tfrac{1}{k^2}\nabla'_\wedge J] \\ a_M(lm) &= \frac{-k^2}{\sqrt{l(l+1)}} \int d^3r'\, j_l(kr')\, Y^*_{lm}(\theta'\phi')\, L'\cdot(J + \nabla_\wedge \mathcal{M}) \end{aligned}}$$

Transform integrands

$$L\cdot A = i\nabla\cdot(r_\wedge A), \quad L\cdot(\nabla_\wedge A) = i\nabla^2(r\cdot A) - \frac{i}{r}\frac{\partial}{\partial r}(r^2\nabla\cdot A)$$

$$a_E = \frac{ik^3}{\sqrt{l(l+1)}} \int d^3r'\, Y^*_{lm} j_l \{ i\nabla\cdot(r_\wedge \mathcal{M}) + \frac{1}{k^2}[i\nabla^2(r\cdot J) - \frac{i}{r}\frac{\partial}{\partial r} r^2 \nabla\cdot J] \}$$

$$\nabla \cdot J = i\omega\rho$$

$$a_E = -\frac{k^3}{\sqrt{l(l+1)}} \int d^3r' \, j_l \, Y^*_{lm} \left\{ \vec{\nabla} \cdot (r \wedge \mathcal{M}) + \frac{1}{k^2} \nabla^2 (\vec{r} \cdot \vec{J}) - \frac{i\omega}{r} \frac{\partial}{\partial r} (r^2 \rho) \right\} \qquad \omega = \frac{c}{k}$$

$$\nabla^2 j_l(kr') Y^*_{lm}(\theta'\phi') = -k^2 j_l(kr') Y^*_{lm}(\theta'\phi')$$

integrate 2nd term by parts twice
integrate 3rd term by parts

$$\boxed{a_E(l,m) = -\frac{ik^2}{\sqrt{l(l+1)}} \int Y^*_{lm} \left\{ c\rho \frac{\partial}{\partial r}[r j_l(kr)] + ik \, \vec{r} \cdot \vec{J} \, j_l(kr) - ik \, \nabla \cdot (r \wedge \mathcal{M}) \, j_l(kr) \right\} d^3x}$$

exact valid for any sized source distribution. Only assumption is observation point outside of source. Valid for any value of kd or $\frac{d}{\lambda}$

Likewise

$$a_M(l,m) = \frac{-ik^2}{\sqrt{l(l+1)}} \int Y^*_{lm} \left\{ \nabla \cdot (r \wedge J) \, j_l + (\nabla \cdot \mathcal{M}) \frac{\partial}{\partial r}(r j_l) - k^2 (r \cdot \mathcal{M}) \, j_l \right\} d^3x$$

If $kd \ll 1$ than expand spherical bessel func.

$$j_l \sim \frac{(kr)^l}{(2l+1)!!}$$

$$\frac{\partial}{\partial r} r j_\ell \sim (\ell+1) \frac{k^\ell r^\ell}{(2\ell+1)!!}$$

1st term involves

$$a_E = -i k^2 c \sqrt{\frac{\ell+1}{\ell}} \frac{k^\ell}{(2\ell+1)!!} \int r^\ell Y^*_{\ell m} \rho \, d^3x + \ldots$$

$$a_E = \frac{-i c k^{\ell+2}}{(2\ell+1)!!} \left(\frac{\ell+1}{\ell}\right)^{1/2} \left(Q_{\ell m} + Q'_{\ell m}\right)$$

$$Q_{\ell m} = \int r^\ell Y^*_{\ell m} \rho \, d^3x$$ normal multipole moments of charge dist.

$$Q'_{\ell m} = \frac{-ik}{(\ell+1)c} \int r^\ell Y^*_{\ell m} \nabla \cdot (r \wedge \mathcal{M}) \, d^3x$$

induced electric multipole moment from magnetization dist.

Note $\int Y^*_{\ell m} j_\ell \, \vec{r} \cdot \vec{J}$ involves $\int r^{\ell+1} Y^*_{\ell m}$ and this will have one higher power of $kd \ll 1$ compared to term we kept.

Note $Q'_{\ell m}$ is in general smaller than $Q_{\ell m}$

Likewise in long wavelength limit

$$a_M(\ell m) \simeq \frac{i k^{\ell+2}}{(2\ell+1)!!} \left(\frac{\ell+1}{\ell}\right)^{1/2} \left(M_{\ell m} + M'_{\ell m}\right)$$

$$M_{\ell m} = -\frac{1}{\ell+1} \int r^\ell Y^*_{\ell m} \nabla \cdot (r \wedge J) \, d^3x$$

$$M'_{lm} = -\int r^l Y^*_{lm} (\nabla \cdot \mathcal{M})\, d^3x$$

For magnetic multipole radiation M and M' can be comparable.

<u>Multipole radiation in Atoms + Nuclei</u>

Use semiclassical arguments (really need QM)

Transition probability Γ (1/mean life) for emission of a photon of energy $\hbar\omega$ is radiated power divided by $\hbar\omega$

$$\Gamma = \frac{P}{\hbar\omega} = \frac{Z_0}{2k^2\hbar\omega} \sum_{lm} \left[|a_E(lm)|^2 + |a_M(lm)|^2 \right]$$

$$= \frac{Z_0}{2k^2\hbar\omega} \frac{c^2 k^{2l+4}}{((2l+1)!!)^2} \frac{l+1}{l} \left| Q_{lm} + Q'_{lm} \right|^2$$

$$+ \quad Q_{lm} + Q'_{lm} \to \frac{1}{c}(M_{lm} + M'_{lm})$$

$$k = \frac{\omega}{c}$$

$$\Gamma = \frac{\omega Z_0 k^{2l}}{2\hbar\left[(2l+1)!!\right]^2} \left(\frac{l+1}{l}\right) \left| Q_{lm} + Q'_{lm} \right|^2$$

Consider a system with effective charge e, mass m and size R

$$Q_{lm} \sim O(e R^l)$$

$$Q'_{lm} \sim \frac{k}{c} R^l \int (\nabla \cdot (r \wedge \mathcal{M}))\, d^3x$$

$$\mathcal{M} \sim \frac{\hbar e}{m} \frac{1}{R^3} \qquad \int_R d^3x\, \mathcal{M} = \vec{m} \text{ dipole moment}$$

$$Q'_{lm} \sim \omega R^{l} \left(\frac{e}{m\,\cancel{R}}\right)$$

$$= Q_{lm} \left(\frac{\hbar\omega}{m\,\cancel{R}c^2}\right)$$

$$|Q'_{lm}| = O\left(eR^{l}\right)\left(\frac{\hbar\omega}{mc^2}\right)$$

For atoms $\hbar\omega \sim eV$ while $m_e c^2 = 511\,keV$

$$\frac{1}{c}\left|M_{lm} + M'_{lm}\right| \sim O\left[\frac{e\hbar}{mc} R^{l-1}\right]$$

$$\frac{\Gamma_M(l)}{\Gamma_E(l)} \sim \left|\frac{\hbar}{mcR}\right|^2 = \left(\frac{\cdot Z_{eff}\alpha}{\cancel{\alpha}}\right)^2 = \left(\frac{Z_{eff}}{137}\right)^2$$

$$R = a_0 / Z_{eff} \qquad \frac{\Gamma(l+1)}{\Gamma(l)} \sim \underset{\text{size}}{O(k^2R^2)} \sim \left(\frac{Z_{eff}}{137}\right)^2$$

$a_0 =$ ~~ab~~ bohr radius $= \dfrac{\hbar}{m_e c\alpha}$

Example $J_i = \frac{3}{2}$ $J_f = \frac{1}{2}$

Can have $l = 1$ or 2

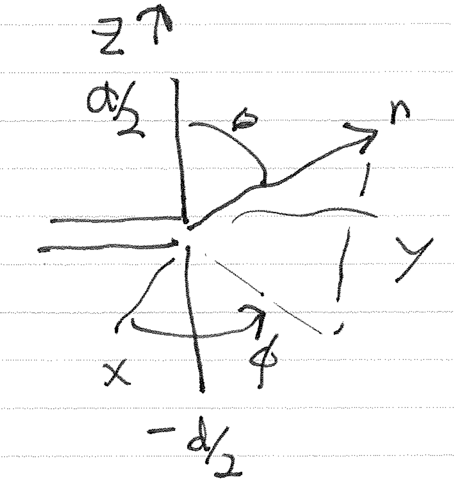
If parity is opp. E1 or M2

E1 will dominate

If parity is same E2 or M1

and they may be comparable.

Multipole radiation from center fed linear antenna



In section 9.4 assumed

$$J(x) = I \sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\delta(y)\hat{z}$$

for $|z| < d/2$

found exact result

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta}\right|^2 \qquad (9.56)$$

Now let us calculate multipole expansion

Rewrite J in spherical coordinates

$$J(\vec{x}) = \hat{r}\,\frac{I(r)}{2\pi r^2}\left[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)\right]$$

for $r < d/2$

$$\rho(\vec{x}) = \frac{1}{i\omega}\frac{\partial I}{\partial r}\left[\frac{\delta(\cos\theta - 1) - \delta(\cos\theta + 1)}{2\pi r^2}\right]$$

$$a_E(lm) = \frac{k^2}{i\sqrt{l(l+1)}}\int Y^*_{lm}\left\{c\rho\frac{\partial}{\partial r}(r j_l) + ik(\vec{r}\cdot\vec{J})\,j_l(kr)\right\}d^3r \qquad 9.167$$

Note $\mathcal{M} \equiv 0$

$$a_M = \frac{k^2}{i\sqrt{l(l+1)}}\int Y^*_{lm}\,\vec{\nabla}\cdot(\vec{r}\wedge\vec{J})\,j_l(kr)\,d^3x \equiv 0 \qquad 9.168$$

since $\vec{J}$ is in $\vec{r}$ direction

No magnetic ~~monopoles~~ multipoles.