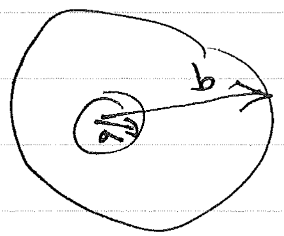
9/28/10

Lec. 9 Boundary Value Problems with Green Functions

Last time G for spherical coord. with $G = 0$ for $x, x' = a$ or b



$$\boxed{G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta'\phi') Y_{lm}(\theta\phi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}')\, G(\vec{x}, \vec{x}')\, d^3x'$$
$$- \frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G}{\partial n'}\, da'$$

Consider potential $\underline{\text{inside}}$ sphere of radius b, (let $a \to 0$)

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=b} = -\frac{4\pi}{b^2} \sum_{lm} Y_{lm}^*(\theta'\phi') Y_{lm}(\theta\phi) \left(\frac{r}{b}\right)^l$$

Now find Φ inside sphere given $\Phi(\vec{x}) = V(\theta, \phi)$ on sphere.

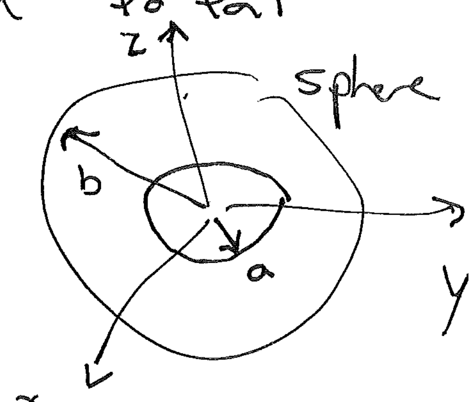
$$\Phi(r,\theta,\phi) = \sum_{lm} \oint_S b^2 \frac{V(\theta'\phi')}{b^2} Y^*_{lm}(\theta'\phi')\, d\Omega' \left(\frac{r}{b}\right)^l Y_{lm}(\theta,\phi)$$

$$da' = b^2\, d\Omega'$$

$$= \sum_{lm}\left[\int_0^{\pi} \sin\theta'\, d\theta' \int_0^{2\pi} d\phi'\; V(\theta'\phi')\, Y^*_{lm}(\theta'\phi')\right] \left(\frac{r}{b}\right)^l Y_{lm}(\theta,\phi)$$

Now conside a nonzero charge density inside volume V. Enough to consider $\Phi \equiv 0$ on surface since we can just add above solution to satisfy $\Phi = V(\theta',\phi')$

Conside hollow grounded sphere of radius b with a concentric ring of radius a and total charge Q



$$\rho(\vec{x}') = \frac{Q}{2\pi a^2}\,\delta(r'-a)\,\delta(\cos\theta')$$ ring in xy plane

indep. of ϕ'

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{2\pi a^2}\right)\int_0^{\infty} r'^2 dr' \int d\cos\theta' \int d\phi'\; \delta(r'-a)\,\delta(\cos\theta')$$

$$4\pi \sum_{lm} \frac{Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)}{2l+1}\, r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

let $r_< = \min(a, r)$

$r_> = \max(a, r)$

$$\int d\phi' \, Y_{lm}(\theta'=0, \phi') = 2\pi \delta_{m0} Y_{l0}(\theta'=0, \phi)$$

$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l(0)$$

$$P_{2n+1}(0) = 0 \, , \qquad P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{2^n n!}$$

Note $P_l(1) = 1$

$l = 2n$

$$\Phi(\vec{x}) = \frac{Q}{8\pi^2 \epsilon_0} 2\pi \sum \frac{4\pi}{(4n+1)} \sqrt{\frac{4n+1}{4\pi}} (-1)^n \frac{(2n-1)!!}{2^n n!}$$

$$Y_{2n0}(\theta, \phi) \, r_<^{2n} \left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}} \right)$$

$$Y_{2n0} = \sqrt{\frac{4n+1}{4\pi}} P_{2n}(\cos\theta)$$

$$\boxed{\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n n!} r_<^{2n} \left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}} \right) P_{2n}(\cos\theta)}$$

Now consider a line charge of total charge Q along z axis

$$\rho(\vec{x}') = \frac{Q}{2b}\frac{1}{2\pi r'^2}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{2b}\right)\frac{1}{2\pi}\int_0^b \frac{r'^2\,dr'}{r'^2}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\cos\theta'$$

$$\sum_{lm}\frac{4\pi}{2l+1}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta,\phi)$$

$$r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)$$

$$Y^*_{lm}(\theta=0\ \phi') = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{2b}\right)\int_0^b dr' \sum_l \left[P_l(1)+P_l(-1)\right] P_l(\cos\theta)$$

$$r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)$$

$$\int_0^b = \left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\int_0^r dr'\,r'^l + r\int_r^b dr'\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right)$$

$$= \frac{2l+1}{l(l+1)}\left[1-\left(\frac{r}{b}\right)^l\right] \qquad = \ln\frac{b}{r} \quad \text{fo, } l=0$$

$$P_\ell(-1) = (-1)^\ell$$

$$\boxed{\Phi(x) = \left(\frac{Q}{4\pi\epsilon_0 b}\right)\left\{\ln\frac{b}{r} + \sum_{j=1}^{\infty}\frac{4j+1}{(2j)(2j+1)}\left[1-\left(\frac{r}{b}\right)^{2j}\right] P_{2j}(\cos\theta)\right\}}$$

note pot. diverges along z axis

Surface charge density

$$\sigma(\theta) = \epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=b} = -\frac{Q}{4\pi b^2}\left[1 + \sum_{j=1}^{\infty}\left(\frac{4j+1}{2j+1}\right)P_{2j}(\cos\theta)\right]$$

Integrate $2\pi\int_{-1}^{1} r^2\sigma(\theta)\Big|_{r=b} \, d\cos\theta = -\frac{Q}{4\pi}4\pi = -Q$

Expansion of G in Cylindrical coord.

$$\nabla_x^2 G(x,x') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z')$$

$$\nabla_x^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

Write Delta functions

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} = \frac{1}{\pi}\int_{0}^{\infty} dk \cos[k(z-z')]$$

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

Expand G

$$G(x,x') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\, g_m(k,\rho,\rho')$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)g_m = -\frac{4\pi}{\rho}\delta(\rho-\rho')$$

For $\rho \neq \rho'$ have

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)g_m = 0$$

let $x' = k\rho$

$$\left[\frac{1}{x'}\frac{d}{dx'}\left(x'\frac{d}{dx'}\right) - \left(1+\frac{m^2}{x'^2}\right)\right]g = 0$$

Bessel's eq.

$$\frac{d^2 g}{dx^2} + \frac{1}{x}\frac{d}{dx}g + \left(1-\frac{m^2}{x^2}\right)g = 0$$

let $x' = ix$

$$\left\{ -\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial}{\partial x} - \left(1 - \frac{m}{x^2}\right) \right\} g = 0$$

or

$$\frac{\partial^2 g}{\partial x^2} + \frac{1}{x}\frac{\partial g}{\partial x} + \left(1 - \frac{m^2}{x^2}\right) g = 0$$

So solutions are Bessel functions of imaginary arguments

$$I_\nu(x) \equiv i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu(ix)$$

I_ν, K_ν are real for x real and are called ~~imaginary~~ modified Bessel functions.

If J_ν, N_ν osc. than I_ν, K_ν are $\sim$ exp.

Remember for Laplace eq. can only have osc. solutions in at most two directions $\Rightarrow$ and exp. in third direction.

$$I_\nu \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \qquad x \ll 1$$

$$I_\nu \to \frac{1}{\sqrt{2\pi x}}\, e^x \left[1 + O\frac{1}{x}\right] \qquad x \gg 1$$

$$K_\nu \to \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \quad \nu \neq 0 \quad x \ll 1 \quad , \sqrt{\frac{\pi}{2x}}\, e^{-x} \qquad \nu \neq 0 \quad x \gg 1$$

Compare with

$$J_\nu(x) \to \begin{cases} \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu & x \ll 1 \\[2ex] \sqrt{\dfrac{2}{\pi x}}\cos\left[x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right] & x \gg 1 \end{cases}$$

Follows from $I_\nu(x) = i^{-\nu} J_\nu(ix)$ end 9/28/10

So general solution for $g_m(k, \rho, \rho')$ $\rho \neq \rho'$

$$g_m = \begin{cases} A(\rho')\, I_\nu(k\rho) + B(\rho')\, K_\nu(k\rho) & \rho > \rho' \\ A'(\rho')\, I_\nu(k\rho) + B'(\rho')\, K_\nu(k\rho) & \rho < \rho' \end{cases}$$

but must by symmetric in ρ, ρ'
write

$$g_m = \psi_1(k\rho_<)\, \psi_2(k\rho_>)$$

Normalization of product $\psi_1 \psi_2$ determined