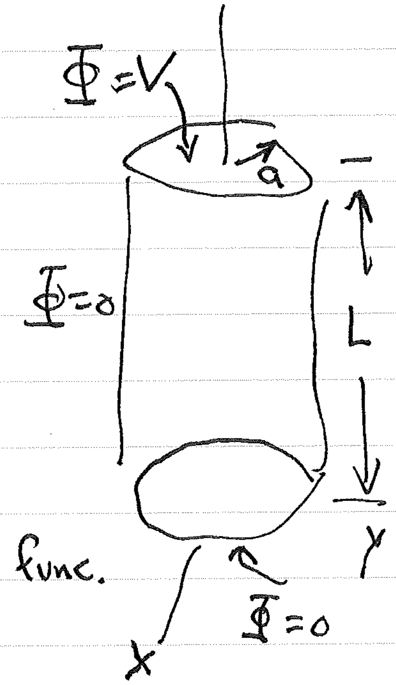
9/23/10

Lec. 8 Green's Functions

Last time cylindrical b.c.



$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left[A_{mn}\sin m\phi + B_{mn}\cos m\phi\right]$$

$$k_{mn} = \frac{x_{mn}}{a} \qquad x_{mn} = n^{th} \text{ zero of } m^{th} \text{ Bessel func.}$$

$$\sinh(k_{mn} z=0) = 0$$

Now satisfy b.c. at $z=L$

$$V(\rho,\phi) = \Phi(\rho,\phi,L) = \sum_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left[A_{mn}\sin m\phi + B_{mn}\cos m\phi\right]$$

Use orthogonality

$$\int_0^a \rho\, J_\nu\!\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\!\left(x_{\nu n}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})\,\delta_{nn'}$$

$$A_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(k_{mn}a)\sinh(k_{mn}L)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, V(\rho,\phi) J_m(k_{mn}\rho)\sin m\phi$$

$$B_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(k_{mn}a)\sinh k_{mn}L} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, V\, J_m(k_{mn}\rho)\cos m\phi$$

Note for $m=0$ use $\frac{1}{2} B_{on}$

If $a \Rightarrow \infty$ orthogonality becomes

$$\int_0^\infty x\,dx\, J_\nu(kx)\, J_\nu(k'x) = \frac{1}{k}\,\delta(k'-k)$$

and sum ove $n \Rightarrow \int dk$

We will sea later spherical Bessel functions

$$j_\ell(z) \equiv \sqrt{\frac{\pi}{2z}}\; J_{\ell+\frac{1}{2}}(z)$$

Sphercal Bessel func. are found in Chap. 9

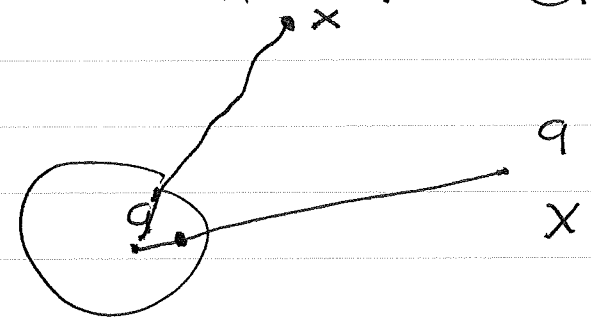
$$\int_0^\infty r^2\, j_\ell(kr)\, j_\ell(k'r)\, dr = \frac{\pi}{2k^2}\,\delta(k-k')$$

Now conside problems with both a distribution of charge and boundary conditions. Solve with Green functions

Conside spherical coordinates. No boundary surfaces except at ∞

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\, \frac{r_<^{\ell}}{r_>^{\ell+1}}\, Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

Now consider exterior problem with a spherical boundary at $r=a$. Use image form of Green function (2.16)



eq. (2.16)

$$G(x,x') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$

This vanishes if $|\vec{x}|=a$ or $|\vec{x}'|=a$

Expand both terms in multipole exp.

$$G(x,x') = 4\pi \sum_{lm} \frac{1}{2l+1} \left\{ \frac{r_<^l}{r_>^{l+1}} - \left(\frac{a}{r'}\right) \left[\frac{a^2}{r'^2} r'\right]^l \frac{1}{r^{l+1}} \right\} Y^*_{lm} Y_{lm}$$

Note $\underbrace{\frac{a^2}{x'}}_{r_<} < \underbrace{|\vec{x}|}_{r_>}$

$$G(x,x') = 4\pi \sum_{lm} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1} \right] Y^*_{lm} Y_{lm}$$

$\frac{a^2}{rr'}$

$$\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1} = \frac{r^{\ell}}{r'^{\ell+1}} - \frac{a^{2\ell+1}}{r^{\ell+1} r'^{\ell+1}} \qquad r<r'$$

$$= \frac{1}{r'^{\ell+1}}\left[r^{\ell} - \frac{a^{2\ell+1}}{r^{\ell+1}}\right] \qquad r<r'$$

$$= \frac{1}{r^{\ell+1}}\left[r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right] \qquad r'<r$$

If r or $r' = a$ $\quad G=0$ ✓

Factor is symmetric in r, r'
viewed a a function of r for fixed r' we have a linear combination of solutions r^{ℓ} or $r^{-(\ell+1)}$

We note that $\frac{\partial}{\partial r}$ of above is discont. at $r=r'$.

Green function with Dirichlet B.C. satisfies

$$\nabla_x^2 \, G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\,\delta(r-r')\,\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

Use completeness relation $\sum_{\ell m} Y^*_{\ell m}(\theta'\phi')\,Y_{\ell m}(\theta\phi) = \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\sum_{lm} Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)$$

Write for G considered function of $\vec{x}$

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r|r',\theta'\phi')\, Y_{lm}(\theta\phi)$$

$$\nabla_x^2 G = -\frac{4\pi}{r^2}\delta(r-r')\sum_{lm} Y^*_{lm}(\theta'\phi')Y_{lm}(\theta\phi)$$

so try

$$A_{lm}(r|r',\theta'\phi') = g_l(r r')\, Y^*_{lm}(\theta'\phi')$$

$$\nabla_x^2\, g_l(r,r')\, Y_{lm}(\theta,\phi) = \left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\, g_l(r,r')) - \frac{l(l+1)}{r^2} g_l(r,r')\right\} Y_{lm}$$

$$= -\frac{4\pi}{r^2}\delta(r-r')\, Y_{lm}(\theta\phi)$$

g_l satisfies

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r g - \frac{l(l+1)}{r^2} g = 0 \qquad \text{for } r \neq r'$$

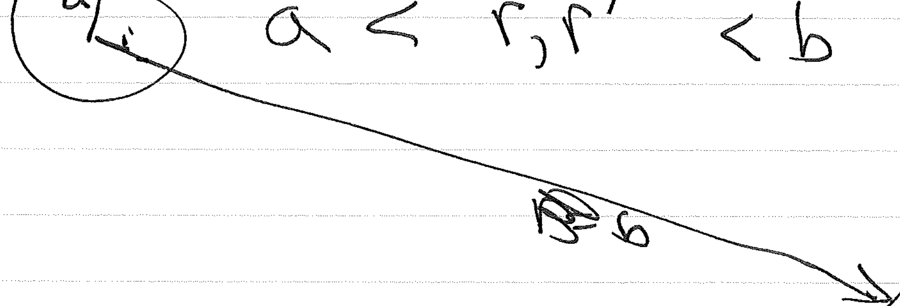
$\Rightarrow$

$$g_l(r,r') = \begin{cases} A\, r^l + B\, r^{-(l+1)} & r<r' \\ A'\, r^l + B'\, r^{-(l+1)} & r>r' \end{cases}$$

A, B, A', B' are functions of r'

Choose to satisfy b.c., symmetry in r, r' and $\delta(r-r')$.

Consider problem with b.c. on $r=a$ and $r=b$



$a < r, r' < b$

$$g_l(r,r') = \begin{cases} A\left(r^l - \dfrac{a^{2l+1}}{r^{l+1}}\right) & r<r' \\ B'\left(\dfrac{1}{r^{l+1}} - \dfrac{r^l}{b^{2l+1}}\right) & r'<r \end{cases}$$

Must be symmetric in r, r' A, B' functions of r' so that

$$\Rightarrow \quad g_l(r,r') = C\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)$$

integrate

$$\int_{r'-\epsilon}^{r'+\epsilon} dr\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rg) - \cancel{\frac{l(l+1)}{r^2}g}\right] = -\frac{4\pi}{r^2}\int_{r'-\epsilon}^{r'+\epsilon}\delta(r-r')\,dr$$

$$\frac{1}{r'}\left[\frac{\partial}{\partial r} rg\Big|_{r'+\epsilon} - \frac{\partial}{\partial r}(rg)\Big|_{r'-\epsilon}\right] = -\frac{4\pi}{r'^2}$$

$$\frac{\partial}{\partial r}(rg)_{r'+\epsilon} = C \frac{\partial}{\partial r}\left(\frac{1}{r^{\ell}} - \frac{r^{\ell+1}}{b^{2\ell+1}}\right)\left(r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right)$$

$$= C\left[(-\ell)\frac{1}{r^{\ell+1}} - (\ell+1)\frac{r^{\ell}}{b^{2\ell+1}}\right]\left[r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right]$$

set $r = r'$

$$= -C\left[\frac{\ell}{r'} + (\ell+1)\frac{r'^{2\ell}}{b^{2\ell+1}}\right]\left[1 - \left(\frac{a}{r'}\right)^{2\ell+1}\right]$$

$$= -\frac{C}{r'}\left[\ell + (\ell+1)\left(\frac{r'}{b}\right)^{2\ell+1}\right]\left[1 - \left(\frac{a}{r'}\right)^{2\ell+1}\right]$$

$$\frac{\partial}{\partial r}(rg)_{r'-\epsilon} = C\left(\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{b^{2\ell+1}}\right)\frac{\partial}{\partial r}\left(r^{\ell+1} - \frac{a^{2\ell+1}}{r^{\ell}}\right)$$

$$= \frac{C}{r'}\left[\ell+1 + \ell\left(\frac{a}{r'}\right)^{2\ell+1}\right]\left[1 - \left(\frac{r'}{b}\right)^{2\ell+1}\right]$$

$$\frac{\partial}{\partial r}(rg)_{r'+\epsilon} - \frac{\partial}{\partial r}(rg)_{r'-\epsilon} = -\frac{4\pi}{r'}$$

$$-\frac{C}{r'}\left\{\ell - \ell\left(\frac{a}{r'}\right)^{2\ell+1} + (\ell+1)\left(\frac{r'}{b}\right)^{2\ell+1} - (\ell+1)\left(\frac{a}{b}\right)^{2\ell+1}\right\}$$

$$-\frac{C}{r'}\left\{\ell+1 + \ell\left(\frac{a}{r'}\right)^{2\ell+1} - (\ell+1)\left(\frac{r'}{b}\right)^{2\ell+1} - \ell\left(\frac{a}{b}\right)^{2\ell+1}\right\}$$

$$= -\frac{C}{r'}(2\ell+1)\left[1 - \left(\frac{a}{b}\right)^{2\ell+1}\right] = -\frac{4\pi}{r'}$$

$$C = \frac{4\pi}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]}$$

$$\boxed{G(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)}{(2l+1)\left(1 - \left(\frac{a}{b}\right)^{2l+1}\right)} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)}$$

If $a \to 0$ and $b \to \infty$ we recover

$$G \to 4\pi \sum_{lm} \frac{Y^* Y}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}}\right)$$

as before

Solution of Potential Problems with Green Function Expansion

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(x')\, G(x,x')\, d^3x' - \frac{1}{4\pi} \oint_S \Phi(x') \frac{\partial G}{\partial n'}\, da'$$

Solution of Poisson eq with Φ specified on surface.

Conside ~~point~~ potential <u>inside</u> sphere of radius b

$$\frac{\partial G}{\partial n'} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b}$$

$$\frac{\partial}{\partial r'}\left(\frac{1}{r'^{\,l+1}} - \frac{r'^{\,l}}{b^{2l+1}}\right) = -\frac{(l+1)}{r'^{\,l+2}} - l\,\frac{r'^{\,l-1}}{b^{2l+1}}$$

now evaluate at $r'=b$

$$= -2(l+1)\frac{1}{b^{l+2}}$$

$$\frac{\partial G}{\partial n'} = -\frac{4\pi}{b^2}\sum_{lm} Y^*\, Y \left(\frac{r}{b}\right)^l$$

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Solution for Φ inside with $\Phi(\vec{x}) = V(\theta,\phi)$ on the sphere is