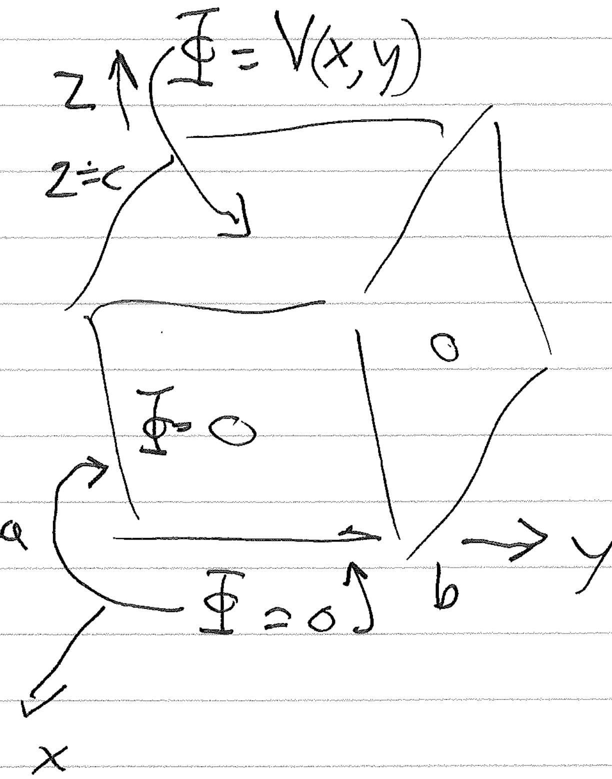
Boundary Value Problems Cont. Lec. 5 9/14/10

Separation of Variables; Laplace Eq in Rectangular Coordinates

<u>Last time</u>

Consider rectangular region with $\Phi = 0$ on 5 ~~ph~~ faces and $\Phi = V(x,y)$ on $z=c$ ~~s~~ face.



$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\Phi = 0$$

$$\Phi(x,y,z) = X_{(x)}\, Y_{(y)}\, Z_{(z)}$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\gamma^2} = \frac{1}{\Phi}\nabla^2\Phi = 0$$

$$\Rightarrow \quad \alpha^2 + \beta^2 = \gamma^2$$

$X = \sin \alpha x$ Have b.c. $\Phi(x=0, y, z) = 0$

$Y = \sin \beta y$ $\quad\Phi(x, y=0, z) = 0$

$Z = \sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)$ $\quad\Phi(x, y, z=0) = 0$

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also need

$$\Phi(x=a, y, z) = 0 \Rightarrow \sin \alpha a = 0$$

$$\alpha = \frac{n\pi}{a}$$

$$\Phi(x, y=b, z) = 0 \Rightarrow \beta = \frac{m\pi}{b}, \quad \sin \beta_m b = 0$$

$$\Phi_{nm}(x,y,z) = \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

$$\gamma_{nm} = \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2}$$

To satisfy b.c. on last face

$$\Phi(x,y,z=c) = V(x,y)$$

take linear superposition

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

$$V(x,y) = \sum_{nm} A_{nm} \sin \alpha_n x \sin(\beta_m y) \sinh(\gamma_{nm} c)$$

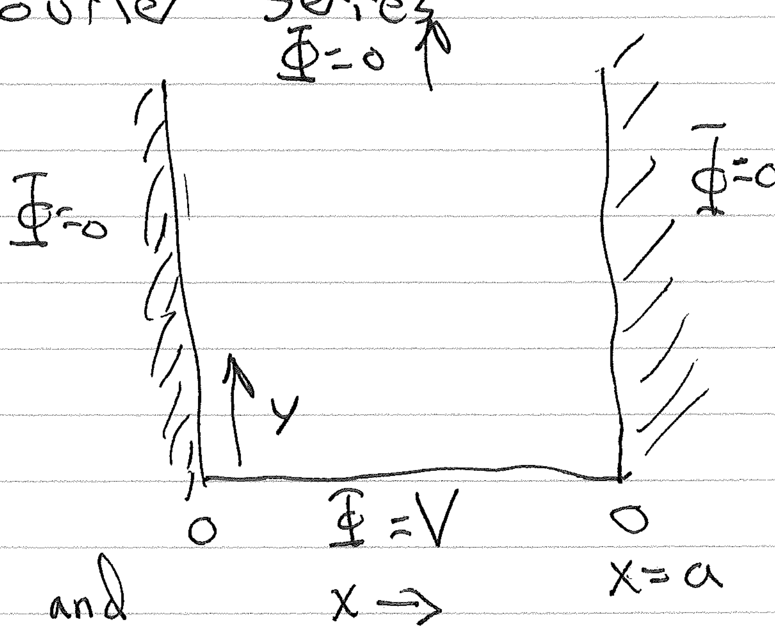
Project out coef. A_{nm} using orthogonality of $\sin \alpha_n x \sin \beta_m y$

$$\boxed{A_{nm} = \frac{4}{ab \sinh \gamma_{nm} c} \int_0^a dx \int_0^b dy\, V(x,y) \sin(\alpha_n x) \sin(\beta_m y)}$$

If $V \neq 0$ on all six sides just take linear superposition of above solution with $\Phi \neq 0$ on each face.

Two Dimensional Potential Problem

Summation of Fourier Series



Two Dim Laplace Eq.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\Phi = 0$$

Can be exp. in one direction and sin / cos in other

$$\Phi \sim e^{\pm i\alpha x}\, e^{\pm \alpha y}$$

for α real or α complex switch $x \leftrightarrow y$

For our problem

$$\Phi(x,y)=0 \qquad x=0,a$$

$$\Rightarrow \text{Choose} \quad \sin \alpha x \quad e^{\pm \alpha y}$$

Also have b.c. $\Phi(y\to\infty)\to 0$
so choose $e^{-\alpha y}$
Need

$$\sin(\alpha a) = \sin n\pi = 0 \Rightarrow \alpha_n = \frac{n\pi}{a}$$

$$\Phi(x,y) = \sum_{n=1}^{a} A_n e^{-\frac{n\pi y}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

A_n are constants. Choose to satisfy b.c. at $y=0$

$$A_n = \frac{2}{a}\int_0^a \Phi(x,0)\sin\frac{n\pi x}{a}\,dx$$

For our problem $\Phi(x,0)=V$ indep. of x

$$A_n = \frac{2V}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)dx = \begin{cases} 4V/\pi n & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

$$\boxed{\Phi(x,y) = \frac{4V}{\pi}\sum_{n\ \text{odd}} \frac{1}{n} e^{-n\pi y/a} \sin\left(\frac{n\pi x}{a}\right)}$$

For large y solution is dominated by $n=1$ term.

$$\Phi(x, y \to \infty) \to \frac{4V}{\pi} e^{-\pi y/a} \sin\left(\frac{\pi x}{a}\right)$$

General result. At large distances y see smoothest component in x.

Can sum full series analytically

$$\sin\left(\frac{n\pi x}{a}\right) = \text{Im}\ e^{i\frac{n\pi}{a}x}$$

so

$$\Phi(x,y) = \frac{4V}{\pi} \text{Im} \sum_{n\,odd} \frac{1}{n} e^{i\frac{n\pi}{a}(x+iy)}$$

Define $\quad Z = e^{i\frac{\pi}{a}(x+iy)}$

$$\Phi = \frac{4V}{\pi} \text{Im} \sum_{n\ odd} \frac{Z^n}{n}$$

Consider $\quad \ln(1+Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 - \frac{1}{4}Z^4$

$$\frac{1}{2}\left(\ln(1+Z) - \ln(1-Z)\right) = \frac{1}{2}\ln\left[\frac{1+Z}{1-Z}\right] = \sum_{n\,odd} \frac{Z^n}{n}$$

$$\Phi = \frac{2V}{\pi} \text{Im}\left\{\ln\left[\frac{1+Z}{1-Z}\right]\right\}$$

$$\Phi = \frac{2V}{\pi} \, \mathrm{Im} \ln \frac{(1+z)(1-z^*)}{|1-z|^2}$$

$$= \frac{2V}{\pi} \left\{ \mathrm{Im} \ln \left[1 - |z|^2 + 2i\,\mathrm{Im}\,z \right] - \underbrace{\mathrm{Im} \ln |1-z|^2}_{=\,0} \right\}$$

$$\mathrm{Im} \ln \left[A + iB \right] = \phi \qquad \text{argument}$$

$$A + iB = Re^{i\phi} \qquad \ln Re^{i\phi} = \ln R + i\phi$$

$$\tan \phi = \frac{B}{A} \qquad A = 1 - |z|^2$$

$$B = 2\,\mathrm{Im}\,z$$

$$\Phi = \frac{2V}{\pi} \tan^{-1} \left[\frac{2\,\mathrm{Im}\,z}{1-|z|^2} \right]$$

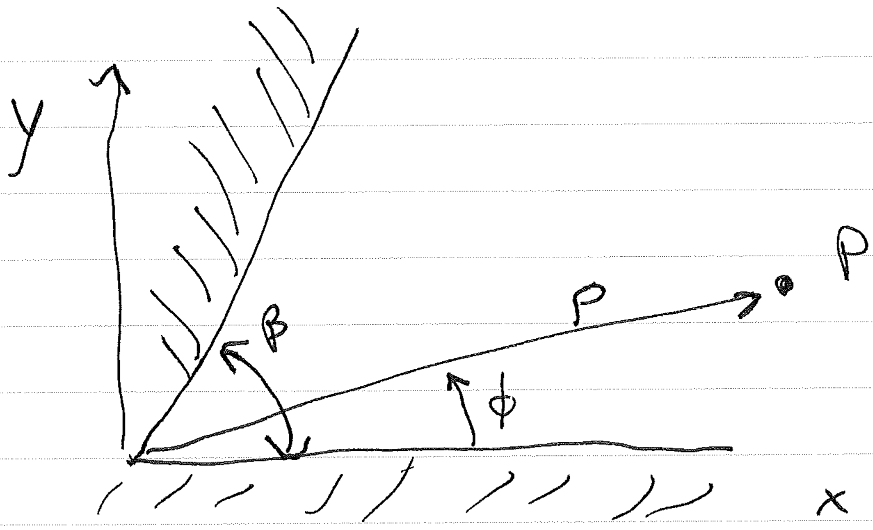
$$\mathrm{Im}\,z = \mathrm{Im}\, e^{i\frac{\pi}{a}(x+iy)} = \sin\left(\frac{\pi x}{a}\right) e^{-(\pi y/a)}$$

$$|z| = e^{-\frac{\pi}{a}y}$$

$$\Phi = \frac{2V}{\pi} \tan^{-1} \underbrace{\frac{2 \sin \frac{\pi x}{a}}{e^{\pi y/a}\left(1 - e^{-\frac{2\pi y}{a}}\right)}}$$

$$\boxed{\Phi(x,y) = \frac{2V}{\pi} \tan^{-1} \left[\frac{\sin \pi x/a}{\sinh \pi y/a} \right]}$$

Two-Dimensional Corners



B. Conditions $\Phi(\rho,\phi) = V$ for $\phi = 0, \beta$

General solution of Laplace Eq. in polar coord.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi = 0$$

$$\Phi = R(\rho)\,\Psi(\phi)$$

$$\underbrace{\frac{\rho}{R}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho}}_{+\nu^2} + \underbrace{\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial\phi^2}}_{-\nu^2} = 0$$

$$R(\rho) = a\rho^{\nu} + b\rho^{-\nu} \qquad \nu = 0 \Rightarrow \quad R = a_0 + b_0 \ln\rho$$

$$\Psi = A\cos\nu\phi + B\sin\nu\phi \qquad \Psi = A_0 + B_0\phi$$

If

$$\Psi(\phi=0) = \Psi(\phi=\beta) = 0$$

$$\Rightarrow \Psi = \cancel{B} \sin\left(\frac{m\pi}{\beta}\phi\right) \qquad m=1,2,3,\ldots$$

$$\boxed{\Phi(\rho,\phi) = V + \sum_{m=1}^{\infty} a_m \rho^{m\pi/\beta} \sin(m\pi\phi/\beta)}$$

The a_m depend on Φ ~~far~~ at ~~distances~~ large distances from corner

For $\rho \to 0$ very near the corner

$$\Phi \xrightarrow[\rho\to 0]{} V + a_1 \rho^{\pi/\beta} \sin(\pi\phi/\beta)$$

Electric field

$$E_\rho = -\partial\Phi/\partial\rho = -\frac{\pi a_1}{\beta} \rho^{\frac{\pi}{\beta}-1} \sin(\pi\phi/\beta)$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} = -\frac{\pi a_1}{\beta} \rho^{\frac{\pi}{\beta}-1} \cos(\pi\phi/\beta)$$

Surface charge density

$$\sigma = \epsilon_0 E_\phi = \epsilon_0 \vec{E}\cdot\hat{n} \simeq -\frac{\epsilon_0 \pi a_1}{\beta} \rho^{\frac{\pi}{\beta}-1}$$

Discussion of lightning rods.
Note that the charge density and electric field $E \sim \rho$ diverges as one approaches a sharp corner. A rod with a rounded end of diameter d, or a thin sheet of thickness d will have an electric field

$$E \sim \frac{1}{d^{1/2}}$$

In vacuum, very large fields are possible but in air there will be a breakdown for E

$$E \gtrsim 2.5 \times 10^{6} \ V/m$$

(at normal T, P).

Thunderstorms can have large potential differences between clouds and ground. A grounded sharp conductor can generate a large E and lead to a breakdown of the air.

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