9/9/10

Lec. 4 Green Function for Sphere

Read Chap. 2. PS #1 due 9/14/10

Used images for pt. charge near a conducting sphere

$$\Phi(\vec{x}) = \frac{q/4\pi\epsilon_0}{|\vec{x}-\vec{y}|} + \frac{q'/4\pi\epsilon_0}{|x-y'|}$$

$$q' = -\frac{a}{y}q \qquad y' = \frac{a^2}{y}$$

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{x}-\vec{y}|} - \left(\frac{a}{y}\right)\frac{1}{|\vec{x}-\frac{a^2}{y^2}\vec{y}|}\right\}$$

Consider

$$G(x,x') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'|\vec{x}-\frac{a^2}{x'^2}\vec{x}'|}$$

$$\Phi(\vec{x}) = \int \frac{d^3x'}{4\pi\epsilon_0}\rho(x')\, G(x,x')$$

with $\rho(x') = q\,\delta(\vec{x}'-\vec{y})$

Note that $G(x,x')$ has form

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(x,x')$$

with $\nabla^2 F(x,x') = 0$ since $\vec{x} \neq a^2 \vec{x}'/x'^2$

$$G(\vec{x},\vec{x}') = \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{a}{x'\left(x^2+\frac{a^4}{x'^2}-2\frac{a^2 x\cdot x'}{x'^2}\right)^{1/2}}$$

$$= \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{a}{(x'^2x^2+a^4-2a^2xx'\cos\gamma)^{1/2}}$$

$$= \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{x'^2x^2}{a^2}+a^2-2xx'\cos\gamma\right)^{1/2}}$$

$$\Rightarrow \quad G(x,x') = G(x',x)$$

also

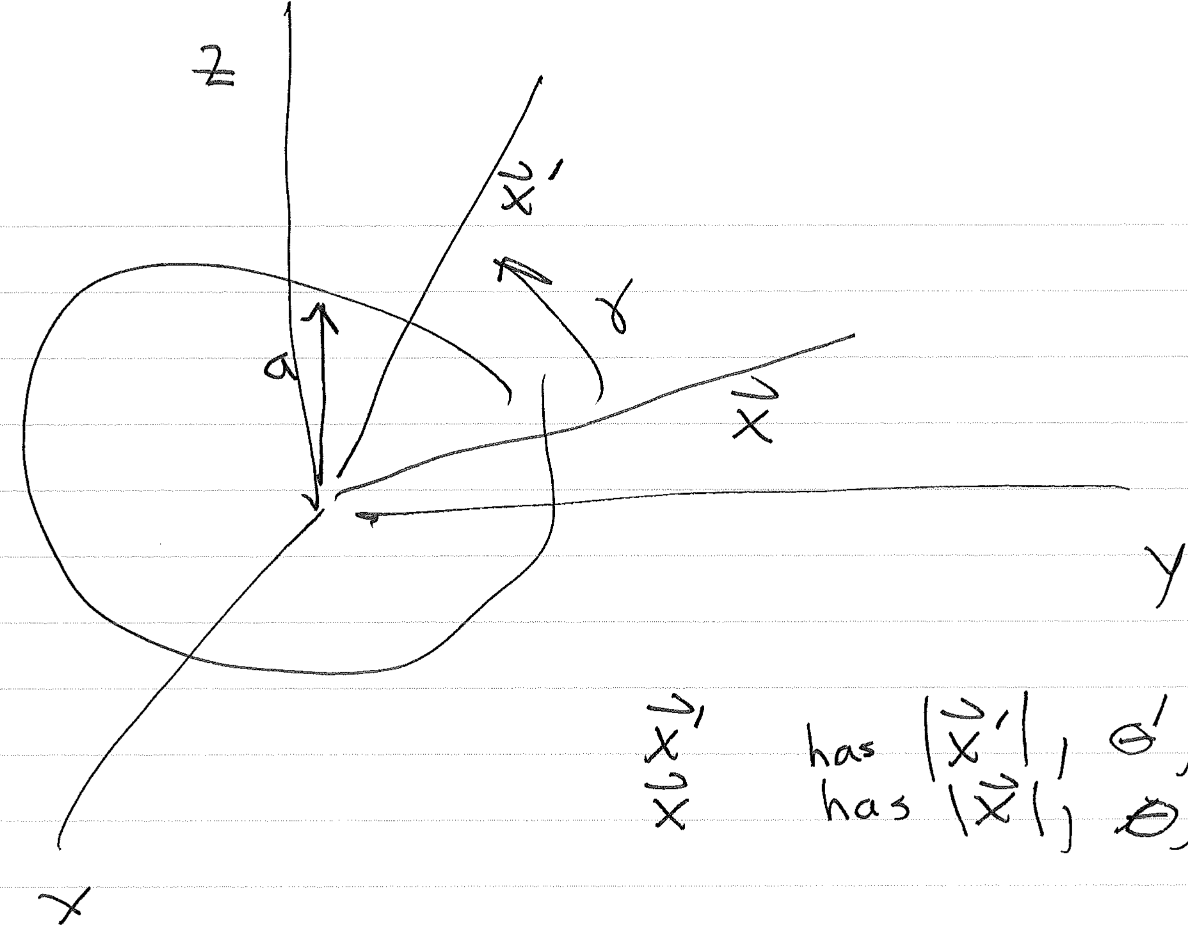
$$G(|\vec{x}|=a, x') = G(\vec{x}, |\vec{x}'|=a) = 0$$

$$G(\vec{a}, x') = \frac{1}{a^2+x'^2-2ax'\cos\gamma} - \frac{1}{(x'^2+a^2-2ax'\cos\gamma)^{1/2}} = 0 \quad \checkmark$$

For solution of Poisson eq. with given dirichlet b.c. need

$$\left.\frac{\partial G}{\partial n'}\right|_{x'=a} = \frac{(x^2-a^2)}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

n' is unit normal directed outward from region of interest $\Rightarrow$ inward to origin



$\vec{x}'$ has $|\vec{x}'|, \theta', \phi'$
$\vec{x}$ has $|\vec{x}|, \theta, \phi$

solution of Laplace eq. outside a sphere with pot. specified on sphere is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x' \, G(x,x')\,\rho(x')$$

$$+ \frac{1}{4\pi}\int \Phi(a,\theta',\phi') \frac{a^2 (x^2-a^2)\, d\Omega'}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

$d\Omega' = \sin\theta' d\theta' d\phi'$

note $\cos\gamma = \dfrac{\vec{x}\cdot\vec{x}'}{x\,x'} = \dfrac{x_1 x_1' + x_2 x_2' + x_3 x_3'}{x\,x'}$

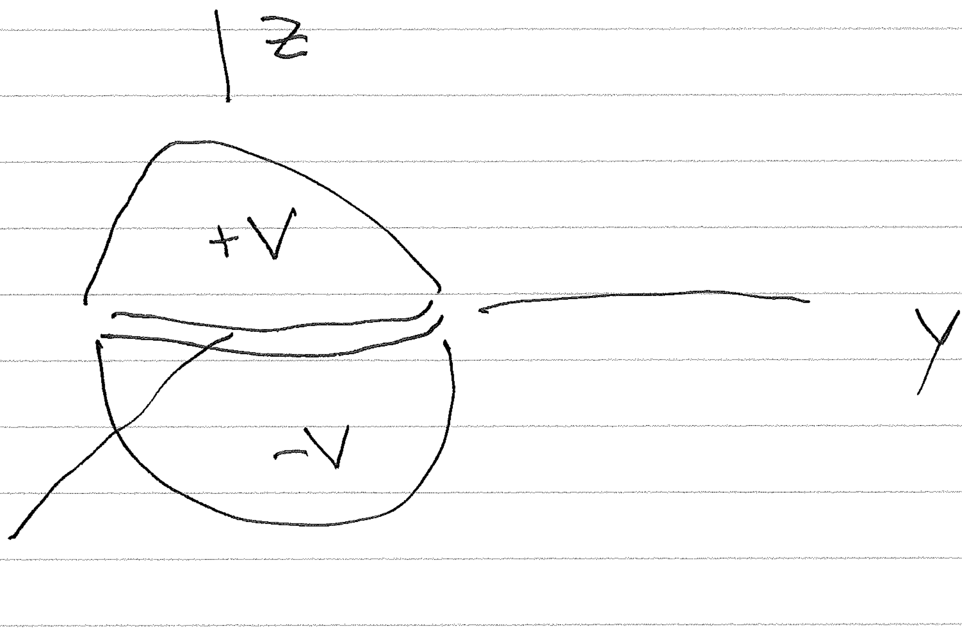
$x_3 = x\cos\theta \qquad x_3' = x'\cos\theta'$

$x_1 = x\sin\theta\cos\phi$

$x_2 = x\sin\theta\sin\phi$

$\cos\gamma = \sin\theta\cos\phi\sin\theta'\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi'$
$+ \cos\theta\cos\theta'$

Conducting sphere with Hemispheres at different potentials



$$\Phi(x\theta\phi) = \frac{V}{4\pi}\int_0^{2\pi} d\phi' \left\{ \int_0^1 d\cos\theta' - \int_{-1}^0 d\cos\theta' \right\} \frac{a^2(x^2-a^2)}{(a^2+x^2-2ax\cos\gamma)^{3/2}}$$

let $\theta' \to \pi - \theta'$, $\phi' \to \phi + \pi$

$$\cos\gamma \to \sin\theta \, \overbrace{\sin(\pi-\theta')}^{\sin\theta'} \left(\cos\phi \underbrace{\cos(\phi'+\pi)}_{-\cos\phi'} + \sin\phi \underbrace{\sin(\phi'+\pi)}_{-\sin\phi'}\right) + \cos\theta \underbrace{\cos(\pi-\theta')}_{-\cos\theta'}$$

$$\to -\cos\gamma$$

$$\Phi(x\,\theta\,\phi) = \frac{V a(x^2-a^2)}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \left[\frac{1}{(a^2+x^2-2ax\cos\gamma)^{3/2}} - \frac{1}{(a^2+x^2+2ax\cos\gamma)^{3/2}}\right]$$

Now let $\theta = 0$ $\quad \cos\gamma = \cancel{\sin\theta}\ \cos\theta'$

$$\Phi(\cancel{z}) = V\left[1 - \frac{z^2-a^2}{z(z^2+a^2)^{1/2}}\right]$$

Expand for large x

$$\alpha = ax/(a^2+x^2) \ll 1$$

$$\Phi(x,\theta,\phi) = \frac{V\ a(x^2-a^2)}{4\pi\ (x^2+a^2)^{3/2}} \int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \left[\frac{1}{(1-2\alpha\cos\gamma)^{3/2}} - \frac{1}{(1+2\alpha\cos\gamma)^{3/2}}\right]$$

expand in powers of α

$$[\ ] \approx 6\alpha\cos\gamma + 35\alpha^3\cos^3\gamma + O(\alpha^5)$$

$$\int_0^{2\pi}\phi' \int_0^1 d\cos\theta' \cos\gamma = \pi\cos\theta$$

$$\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \cos^3\gamma = \frac{\pi}{4}\cos\theta\,(3-\cos^2\theta)$$

$$\Phi \simeq \frac{V a (x^2 - a^2)}{4\pi (x^2+a^2)^{3/2}} \Big\{ 6\alpha\pi\cos\theta + 35\alpha^3 \frac{\pi}{4}\cos\theta\,(3-\cos^2\theta) + \ldots \Big\}$$

$$\alpha = \frac{a x}{a^2+x^2}$$

$$\boxed{\Phi \simeq \frac{3}{2} V \frac{a^2 x (x^2-a^2)}{(x^2+a^2)^{5/2}} \cos\theta \left[1 + \frac{35}{24}\frac{a^2x^2}{(a^2+x^2)^2}(3-\cos^2\theta) + \ldots \right]}$$

$$x \gg a \qquad x^2 - a^2 \simeq x^2$$
$$(x^2+a^2) \simeq x^2$$

$$\Phi \simeq \frac{3}{2} V \frac{a^2}{x^2} \cos\theta$$

Note

$$\frac{x^3(x^2-a^2)}{x^5\left(1+\frac{5}{2}\frac{a^2}{x^2}\right)} \simeq 1 - \frac{7}{2}\frac{a^2}{x^2}$$

$$\Phi \simeq \frac{3}{2} V \frac{a^2}{x^2}\left(1 - \frac{7a^2}{2x^2}\right)\left[\cos\theta + \frac{35}{24}\frac{a^2}{x^2}(3-\cos^2\theta) + O\left(\frac{a^4}{x^4}\right)\right]$$

$$\simeq \frac{3Va^2}{2x^2}\Big\{\cos\theta + \frac{a^2}{x^2}\left[-\frac{7}{2} + \frac{3\cdot 35}{24} - \frac{35}{24}\cos^2\theta\right]\cos\theta\Big\}$$

$$\frac{21}{24}\cos\theta - \frac{35}{24}\cos^3\theta$$

$$-\frac{7}{12}\frac{a^2}{x^2}\left[-\frac{3}{2}\cos\theta + \frac{5}{2}\cos^3\theta\right]$$

$$\boxed{\Phi \cong \frac{3Va^2}{2x^2}\left\{\cos\theta - \frac{7}{12}\frac{a^2}{x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots\right\}}$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$$

See p 97 Legendre Polynomials

Orthogonal Function Expansions

Consider functions $U_n(t)$ on interval (a,b)

$$\int_a^b U_n^*(t)\,U_m(t)\,dt = \delta_{nm}$$

Legendre functions on interval $(-1,1)$

$$\int_{-1}^{1} dt\, P_1^2(t) = \int_{-1}^{1} dt\, t^2 = \left.\frac{t^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

So

$$U_\ell \equiv \sqrt{\frac{2\ell+1}{2}}\, P_\ell$$

$$\int_{-1}^{1} dt\, U_1^2 = \frac{3}{2}\int_{-1}^{1} dt\, t^2 = 1$$

$$\int_{-1}^{1} dt\, P_1\, P_3 = \int_{-1}^{1} dt\, t\left(\frac{5}{2}t^3 - \frac{3}{2}t\right)$$

$$= \int_{-1}^{1} dt\left(\frac{5}{2}t^4 - \frac{3}{2}t^2\right) = 0$$

$$\int_{-1}^{1} dt\, U_3^2 = \frac{7}{2}\int dt\left(\frac{5}{2}t^3 - \frac{3}{2}t\right)^2$$

$$= \frac{7}{2}\int_{-1}^{1} dt\left[\frac{25}{4}t^6 - \frac{15}{2}t^4 + \frac{9}{4}t^2\right]$$

$$= \frac{7}{2}\left[\frac{25}{4}\,\frac{2}{7} - \frac{15}{2}\,\frac{2}{5} + \frac{9}{4}\,\frac{2}{3}\right]$$

$$= \frac{25}{4} - \frac{21}{2} + \frac{21}{4} = 1 \quad \checkmark$$

Expand an arbitrary function $f(t)$

$$f(t) \rightarrow \sum_{n=1}^{N} a_n U_n(t)$$

multiply by $U_m^*(t)$ and integrate

$$\int_a^b dt\, U_m^*(t)\, f(t) = \sum_{n=1}^{N} a_n \underbrace{\int_a^b U_m^*(t)\, U_n(t)\, dt}_{\delta_{mn}}$$

$$a_m = \int_a^b U_m^*(t)\, f(t)\, dt$$

If $$f(t) = \sum_{n=1}^{\infty} a_n U_n(t)$$

$$f(t) = \sum_{n=1}^{\infty} \int_a^b dt'\, U_n^*(t')\, U_n(t)\, f(t')$$

consider

$$\boxed{\sum_{n=1}^{\infty} U_n^*(t')\, U_n(t) = \delta(t-t')}$$

completeness relation

$$f(t) = \int_a^b dt' \sum_{n=1}^{\infty} U_n^*(t')\, U_n(t)\, f(t')$$

$$= \int_a^b dt'\, \delta(t-t')\, f(t') = f(t) \checkmark$$

<u>Simple example</u> Fourier Series

let interval be $-\frac{a}{2}$ to $a/2$

$$\sqrt{\frac{2}{a}} \sin \frac{2\pi m x}{a}, \quad \sqrt{\frac{2}{a}} \cos \frac{2\pi m x}{a}$$

m is non-negative integer

$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos \frac{2\pi m x}{a} + B_m \sin \frac{2\pi m x}{a} \right]$$

$$A_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x)\, dx \cos(2\pi m x/a)$$

$$B_m = \frac{2}{a} \int f(x)\, dx \sin 2\pi m x/a$$