P506 Lec. 3   Green's Functions

Read  Chap. 1
Homework 1 Due 9/14/10

Last time

\[ \begin{align*}
\Phi &= \int_0^V S \left( \frac{\rho(x')}{R} \right) dx' + \frac{1}{4\pi} \int_S \left[ \frac{\Phi}{R} \frac{\partial}{\partial n'} \frac{1}{R} \right] da'
\end{align*} \]

Used to prove uniqueness of solution

For \( \Phi \) specified on \( S \)

Dirichlet b.c. Neumann b.c.

\[ \Delta = \frac{1}{|x-x'|} = -4\pi \delta(x-x') \]

in general

\[ G(x, x') = \frac{1}{|x-x'|} + F(x, x') \]

\[ \nabla^2 G(x, x') = -4\pi \delta(x-x') \]

and

\[ \nabla^2 F(x, x') = 0 \]

Can choose \( F \) to satisfy b.c.
Use Green's Thm from last lec.

\[ \Psi(x) = \frac{1}{4\pi \varepsilon_0} \int_V \rho(x') G(x,x') \, dx' \]

\[ + \frac{1}{4\pi} \oint_S [G(x,x') \frac{\partial \Phi}{\partial n'} - \Phi(x') \frac{\partial G(x,x')}{\partial n'}] \, da' \]

For Dirichlet b.c. choose \( G \) so that

\[ G(x,x') = 0 \quad \text{for} \quad x' \text{ on } S \]

\[ \Psi(x) = \frac{1}{4\pi \varepsilon_0} \int_V \rho(x') G(x,x') \, dx' \]

\[ - \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} \, da' \]

Given \( \Phi \) on \( S \) and \( G \) that satisfies b.c. \( G(x,x') = 0 \) \( x' \) on \( S \) \( \Rightarrow \) Solve for \( \Psi(x) \) \( \Phi \) for all \( x \) inside \( S \)

For Neuman b.c. little bit tricky because

\[ \oint_S \frac{\partial G}{\partial n'} \, da' = -4\pi \]

so can't make \( \frac{\partial G}{\partial n'} = 0 \) for \( x' \) on \( S \)
The best one can do is try

$$\frac{\partial G(x, x')}{\partial n'} = -\frac{4\pi}{S}$$

$s =$ total area of surface

$$\Phi(x) = \left< \Phi \right> + \frac{1}{S} \int_0^S \rho(x') G(x, x') d^3 x'$$

$$+ \frac{1}{4\pi S} \oint_{\partial S} \Phi G(x, x') da'$$

Need to know average value of $\Phi$ on $S$ often have exterior problem when one surface goes to infinity with very large area on which $<\Phi>=0$

**Electrostatic Potential Energy**

$$W_i = q_i \Phi(x_i)$$

Work needed to bring $q_i$ in from infinity $\Phi(\infty) = 0$ given $\Phi(x)$

$$\Phi(x_i) = \frac{1}{4\pi \epsilon_0} \sum_{j=1}^{N-1} \frac{q_j}{|x_i - x_j|}$$

$$W_i = \frac{q_i}{4\pi \epsilon_0} \sum_{j=1}^{N-1} \frac{q_j}{x_{ij}}$$

$x_{ij} = |x_i - x_j|$
Total pot. E. found by summing over all:

\[ W = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} \frac{q_i q_j}{x_{ij}} \]

Sum over \( i \neq j \) or sum over \( i \neq j \) and multiply by \( \frac{1}{2} \) since \( x_{ij} \) and \( x_{ji} \) pair get counted twice.

\[ W = \frac{1}{8\pi \varepsilon_0} \sum_{i=1, j=1, i \neq j}^{n} \frac{q_i q_j}{x_{ij}} \]

\( i \neq j \) is assumed.

\[ W = \frac{1}{8\pi \varepsilon_0} \int d^3x \int d^3x' \frac{\rho(x) \rho(x')}{r(x, x')} \]

\[ W = \frac{1}{2} \int d^3x \rho(x) \Delta(x) d^3x \]

Now use Poisson eq.

\[ \nabla^2 \Pi = \frac{\rho}{\varepsilon_0} \]

\[ W = -\frac{\varepsilon_0}{2} \int d^3x \nabla^2 \Pi(x) d^3x \]

Integrate by parts \( (\nabla^2 = -\nabla \Pi) \)

\[ W = \frac{\varepsilon_0}{2} \int d^3x \left| \nabla \Pi \right|^2 d^3x \]
can think of energy density in electric field as \( W = \frac{E^2}{2} |E|^2 \)

For a system of \( n \) conductors each with potential \( V_i \) and charge \( Q_i \):

\[
V_i = \sum_{j=1}^{n} p_{ij} Q_j
\]

where \( p_{ij} \) depend on geometry of conductors. This can be inverted

\[
Q_i = \sum_{j=1}^{n} C_{ij} V_j
\]

\( C_{ii} \) are capacitances

\( C_{ij} (j \neq i) \) are coef. of inductances

Capacitance of a conductor is the total charge on the conductor when held at unit potential and all other conductors are at zero potential.

Pot. E. for system of conductors is

\[
W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j
\]
Can think of energy density in electrostatic field as \( \frac{\varepsilon_0 E^2}{2} \).

Consider a system

Variational Approach will be discussed in classical Mech.

Start reading Chap. 2 Boundary Value Problems in Electrodynamics

Discuss general methods for solving boundary value problems (1) Method of images closely related to our formal Green's Func. method. (2) Expansion in orthogonal functions follows from diff. eq. and (3)
Method of images

Simple example. Charge at by an infinite conducting plane at by zero potential (grounded).

\[ \Phi = 0 \]

Equivalent to problem with image charge \( q' \) and \( \Phi = 0 \)

Image charge allows one to satisfy b.c. that \( \Phi = 0 \) on plane.

2nd example: Charge on a grounded at sphere of radius \( a \).
If one image charge works it will be located on a ray from origin to "y

\[ \Phi = \frac{1}{4\pi \epsilon_0} \left( \frac{q}{|x-y|} + \frac{q'}{|x-y'|} \right) \]

Choose \( q' \) and \( |y'| \) so that \( \Phi(x=a) = 0 \)

If \( \hat{n} \) is in direction \( x \)

\[ \Phi \pi \epsilon_0 = - \frac{q}{|x \hat{n} - y n'|} + \frac{q'}{|x \hat{n} - y' n'|} \]

\[ \Phi(x=a) = - \frac{q}{a |n - \frac{y}{a} n'|} + \frac{q'}{y' |n' - \frac{a}{y} n'|} \]

\[ \frac{q}{a} + \frac{q'}{y'} = 0 \quad \frac{y}{a} = \frac{q}{y'} \]

\[ q' = - \frac{a}{y} q \quad y' = \frac{a^2}{y} \]

As \( q \) is brought near sphere \( q' \) grows in magnitude and is brought close to the sphere.

Note

\[ |n - \frac{y}{a} n'| = \left( 1 - \frac{2y}{a} \cdot \frac{n \cdot n'}{y' a} + \frac{y^2}{a^2} \right)^{\frac{1}{2}} \]

\[ |n' - \frac{a}{y} n| = \left( 1 - \frac{2a}{y'} \cdot \frac{n \cdot n'}{y' a} + \frac{a^2}{y'^2} \right)^{\frac{1}{2}} \]
Actual charge density related to derivative of sphere

\[ (E_2 - E_1) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0} \]

and \[ E_i = -\frac{\Phi}{\varepsilon_0} \]

inside sphere\[ E = 0 \]

\[ \sigma = -\varepsilon_0 \frac{\partial E}{\partial x} \bigg|_{x=a} = -\frac{\varepsilon_0}{4\pi a^2} \left( \frac{a}{y} \right) \frac{1 - \frac{a^2}{y^2}}{(1 + \frac{a^2 - 2a \cos \theta}{y^2})^{3/2}} \]

\( \theta \) is angle between \( x \) and \( \hat{y} \)

[Diagram showing a circle with angles labeled]
The force on the charge \( q \) from the charges \( q' \) can be calculated. Just from the image:

\[
|F| = \frac{q q'}{4 \pi \varepsilon_0 |y - y'|^2}
\]

\[
= \frac{q^2}{4 \pi \varepsilon_0 \left( y - \frac{a^2}{y} \right)^2}
\]

Now consider a charged, insulated conducting sphere before had \( q' = -\frac{aq}{y} \) distributed about sphere in such a way that it was in equilibrium with zero net force. Now add extra charge \( q' \) to sphere. Extra charge will be distributed uniformly.

\[
\Phi(x) = \frac{1}{4 \pi \varepsilon_0} \left[ \frac{q}{|x-y|} \frac{-aq}{y} \frac{a^2 y}{y^2} \right] + \frac{Q + \frac{aq}{y} q}{1x}
\]

Potential of extra charge. Equivalent to pt. charge at origin.
Finally, consider a point charge $Q$, near a conducting sphere. The extra charge on the sphere is $Q' = V$.

The electric field at point $P$ is:

$$E(x) = \frac{1}{4\pi \varepsilon_0} \left[ \frac{q}{|x - y|} - \frac{q}{y} \frac{1}{|x - a^2 y|} \right] + \frac{V a}{12 x^2}$$

Conducting sphere in a uniform $E$ field.

$E_0 \propto \frac{2 Q}{4\pi \varepsilon_0 R^2}$

As $R \to \infty$, the charge is

$Q'' = -\frac{a Q}{R}$

$Q' = \frac{a Q}{R}$
Consider

\[ E_s E = \frac{Q}{(r^2 + R^2 + 2rR \cos \theta)^{1/2}} - \frac{Q}{(r^2 + R^2 - 2rR \cos \theta)^{1/2}} \]

\[ - \frac{aQ}{R \left( \frac{r^2 + a^2 + 2aR \cos \theta}{R^2} \right)^{1/2}} + \frac{aQ}{R \left( \frac{r^2 + a^2 - 2aR \cos \theta}{R^2} \right)^{1/2}} \]

Expand $Q \to \infty$ and work to order $\frac{r^2}{R^2}$ as

\[ 4\pi \varepsilon_0 E = -\frac{2Q}{R^3} r \cos \theta + \frac{2Q}{R^2} \frac{a^2}{r^2} \cos \theta + O\left(\frac{1}{R^3}\right) \]

In limit $R \to \infty$, $E_0 = \frac{2Q}{4\pi \varepsilon_0 R^2}$

\[ E = -E_0 \left( \frac{r^3}{r^2} - \frac{a^3}{r^2} \right) \cos \theta \]

First term is pot. fr. unif. E field and 2nd term is pot. from induced charge on sphere.