9/7/10

P506 Lec. 3 Green's Functions

Read Chap. 1

HW 1 Due 9/14/10

last time

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(x')}{R}\, d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'}\frac{1}{R}\right] da'$$

Used to prove uniqueness of solution

if Φ specified on S Dirichlet b.c.

$\frac{\partial \Phi}{\partial n}$ Neumann b.c.

Used

$$\nabla'^2 \frac{1}{|\vec{x}-x'|} = -4\pi\,\delta(x-x')$$

in general

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(x,x')$$

$$\nabla'^2 G(x,x') = -4\pi\,\delta(x-x')$$

and $\nabla'^2 F(x,x') = 0$

Can choose F to satisfy b.c..

Use Green's Thm from last lec.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G(x\, x')\, d^3x'$$

$$+ \frac{1}{4\pi}\oint_S \left[G(x,x')\frac{\partial \Phi}{\partial n'} - \Phi(x')\frac{\partial G(x,x')}{\partial n'} \right] da'$$

For Dirichlet b.c. choose G so that

$$G(x,x') = 0 \quad \text{for } x' \text{ on } S$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G(x,x')\, d^3x'$$

$$- \frac{1}{4\pi}\oint_S \Phi(x')\frac{\partial G}{\partial n'}(x,x')\, da'$$

Given Φ on S and G that satisfies b.c. $G(x,x') = 0$ x' on S $\Rightarrow$ Solve for $\Phi(x)$ ~~on~~ for all x inside S

For Neuman b.c. little bit tricky because

$$\oint_S \frac{\partial G}{\partial n'}\, da' = -4\pi$$

so can't make $\frac{\partial G}{\partial n'} = 0$ for x' on S

best one can do is try

$$\oint_S \frac{\partial G}{\partial n'}(x,x') = -\frac{4\pi}{S}$$

S = total area of surface

$$\Phi(x) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0}\int \rho(x') G_N(x,x') d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial \Phi}{\partial n'} G(x,x')\, da'$$

Need to know average value of Φ on S
often have exterior problem where one surface goes to infinity with very large area on which $\langle\Phi\rangle = 0$

<u>Electrostatic Potential Energy</u>

$$W_i = q_i \Phi(x_i)$$

work needed to bring q_i in from infinity (where $\Phi = 0$) given $\Phi(x)$

$$\Phi(x_i) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n-1} \frac{q_j}{|x_i - x_j|}$$

$$W_i = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{n-1} \frac{q_j}{x_{ij}} \qquad x_{ij} = |\vec{x}_i - \vec{x}_j|$$

Total pot. E. found by summing over all:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{x_{ij}}$$

Sum over $i>j$ or sum over $i \neq j$ and multiply by $\frac{1}{2}$ since x_{ij} and x_{ji} pair get counted twice

$$W = \frac{1}{8\pi\epsilon_0} \sum_i \sum_j \frac{q_i q_j}{x_{ij}}$$

$i \neq j$ is assumed

$$W = \frac{1}{8\pi\epsilon_0} \int d^3x \int d^3x' \frac{\rho(x)\,\rho(x')}{|\vec{x} - \vec{x}'|}$$

$$= \frac{1}{2} \int \rho(x)\, \Phi(x)\, d^3x$$

Now use Poisson eq.

$$\nabla^2 \Phi = \rho / \epsilon_0$$

$$W = -\frac{\epsilon_0}{2} \int \Phi(x) \nabla^2 \Phi(x)\, d^3x$$

Integrate by parts $\quad (\vec{E} = -\nabla\Phi)$

$$W = \frac{\epsilon_0}{2} \int |\nabla \Phi|^2 d^3x = \frac{\epsilon_0}{2} \int |\vec{E}|^2 d^3x$$

can think of energy density in electrostatic field as $w = \frac{\epsilon_0}{2} |\vec{E}|^2$

For a system of n conductors each with potential V_i and charge Q_i

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j$$

where p_{ij} depend on geometry of conductors. This can be inverted

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

C_{ii} are capacitances

C_{ij} $(j \neq i)$ are coef. of inductances

Capacitance of a conductor is the total charge on the conductor when held at unit potential and all other conductors are at zero potential.

Pot. E. for system of conductors is

$$W = \frac{1}{2} \sum_{j=1}^{n} Q_i V_i = \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} V_i V_j$$

~~Can think of energy density in electrostatic field as~~ $\cancel{w = \frac{\epsilon_0}{2} |\vec{E}|^2}$,

~~Consider a system~~

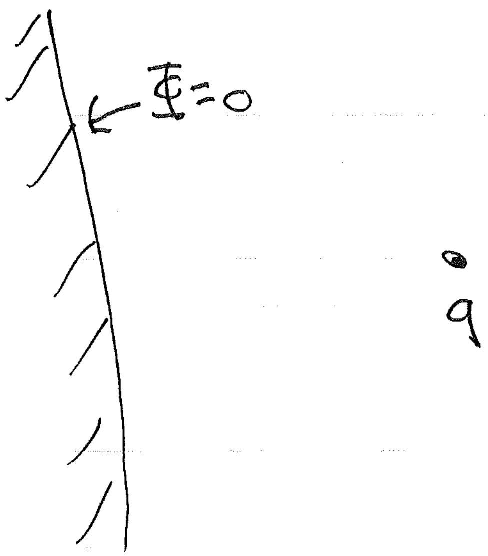
Variational Approach will be discussed in classical Mech.

Start reading Chap. 2
Boundary-Value Problems in Electrostatics: I

Discuss general methods for solving boundary value problems (1) method of images ⟺ closely related to our formal Green's func. method. (2) Expansion ~~as~~ in orthogonal functions follows from dif. eq. ~~and (3)~~

Method of images

Simple example Charge by an infinite conducting plane at zero potential (grounded).



Equivalent to problem with image charge

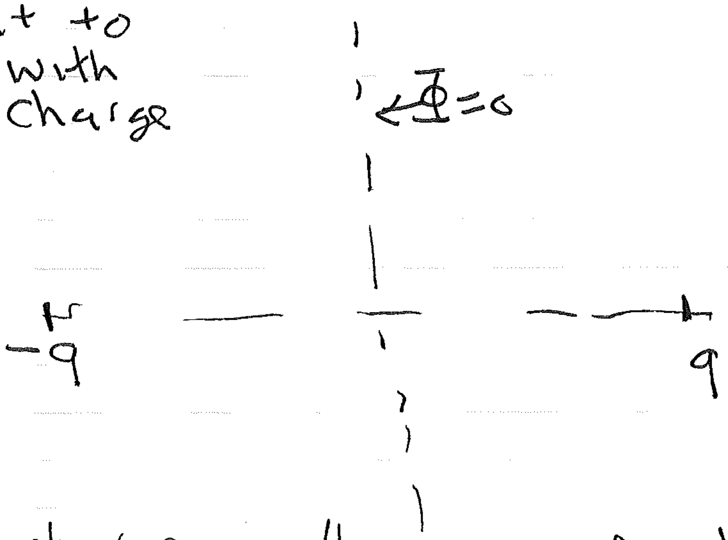
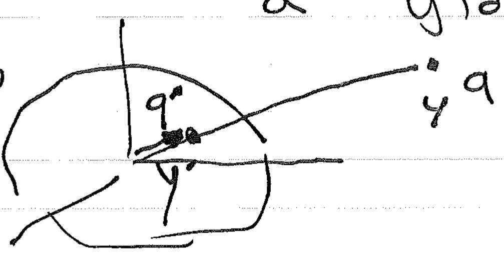


Image charge allows one to satisfy b.c. that $\Phi = 0$ on plane.

2^{nd} example

Charge q at $\vec{y}$ and a grounded sphere of radius a

Conducting sphere radius a



If one image charge works it will be located on ray from origin to y

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right]$$

Chose q' and $|\vec{y}'|$ so that $\Phi(|\vec{x}|=a)=0$

If $\hat{n}$ is in direction $\vec{x}$

$$\Phi\, 4\pi\epsilon_0 = \frac{q}{|x\hat{n} - y n'|} + \frac{q'}{|x\hat{n} - y' n'|}$$

$$4\pi\epsilon_0 \Phi(x=a) = \frac{q}{a|n - \frac{y}{a}n'|} + \frac{q'}{y'|n' - \frac{a}{y'}n|}$$

$$\frac{q}{a} + \frac{q'}{y'} = 0 \qquad \frac{y}{a} = \frac{a}{y'}$$

$$q' = -\frac{a}{y}q \qquad y' = \frac{a^2}{y}$$

as q is brought near sphere q' grows in magnitude and is brought close to the sphere

Note

$$|n - \frac{y}{a}n'| = \left(1 - 2\frac{y}{a}\, n\cdot n' + \frac{y^2}{a^2}\right)^{1/2}$$

$$|n' - \frac{a}{y'}n| = \left(1 - 2\frac{a}{y'}\, n\cdot n' + \frac{a^2}{y'^2}\right)^{1/2}$$

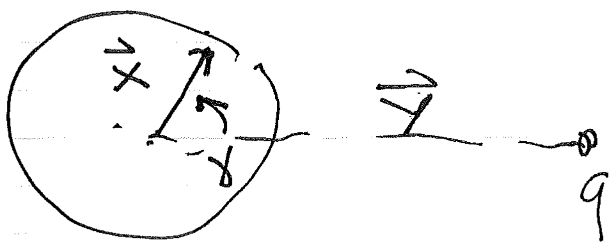
Actual charge density on sphere related to derivative of Φ

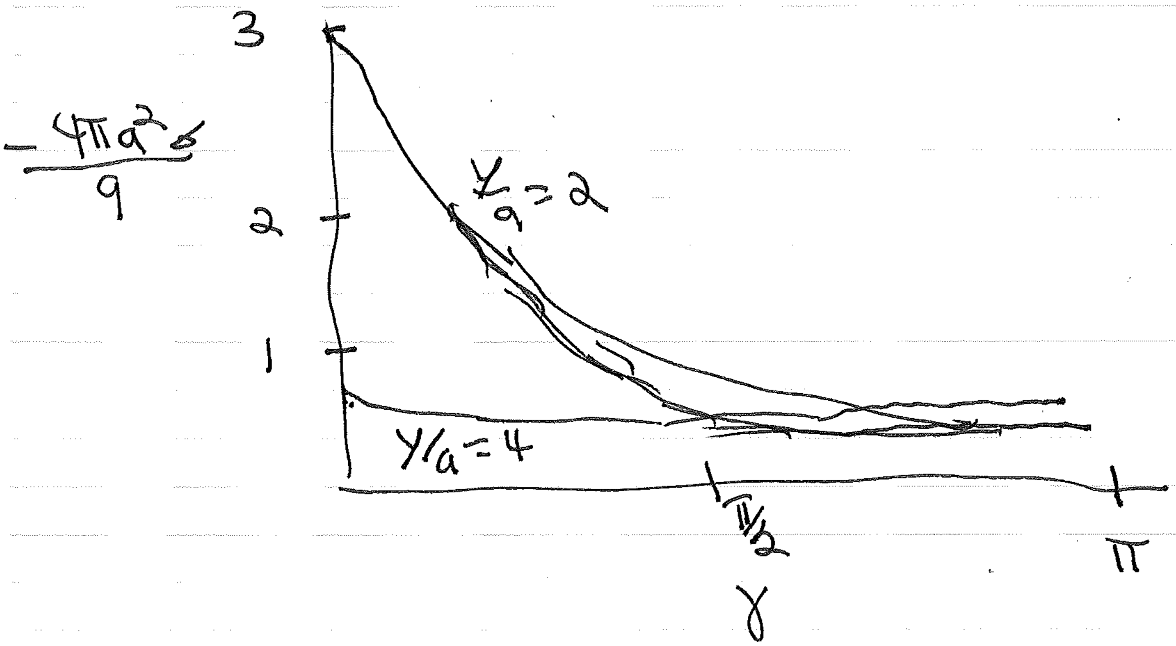
$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \sigma/\epsilon_0$$

and $\vec{E} \cdot \hat{n} = -\frac{\partial \Phi}{\partial n}$ and $E = 0$ inside sphere

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial x}\right|_{x=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1 - a^2/y^2}{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right)^{3/2}}$$

γ is angle between $\vec{x}$ and $\vec{y}$





The force on the charge q can be calculated just from the image charge

$$|F| = \frac{q q'}{4\pi\epsilon_0 |\vec{y} - \vec{y}'|^2}$$

$$= + \frac{a}{y} \frac{q^2}{4\pi\epsilon_0} \frac{1}{\left(y - \frac{a^2}{y}\right)^2}$$

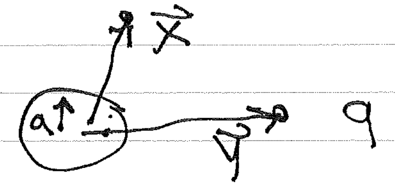
$$= \frac{1}{4\pi\epsilon_0} \frac{q^2 a}{y^3} \frac{1}{\left(1 - \frac{a^2}{y^2}\right)^2}$$

Now consider a charge, insulated conducting sphere.

before had $q' = -\frac{aq}{y}$ distributed about sphere in such a way that it was in equilibrium with zero net force. Now add extra charge $Q - q'$ to sphere. Extra charge will be distributed uniformly so

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|x-y|} - \frac{a}{y}\frac{q}{\left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|} + \frac{Q + \frac{a}{y}q}{|X|}\right]$$

Potential of extra charge. Equivalent to pt. charge at origin

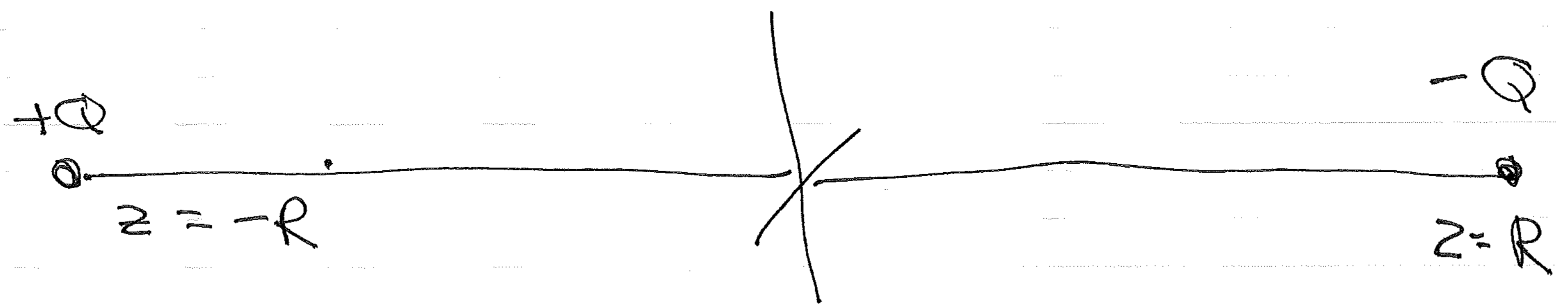


Finally consider pt. charge near conducting sphere held at potential V.

Extra charge on sphere $\frac{Q'}{4\pi\epsilon_0 a} = V$

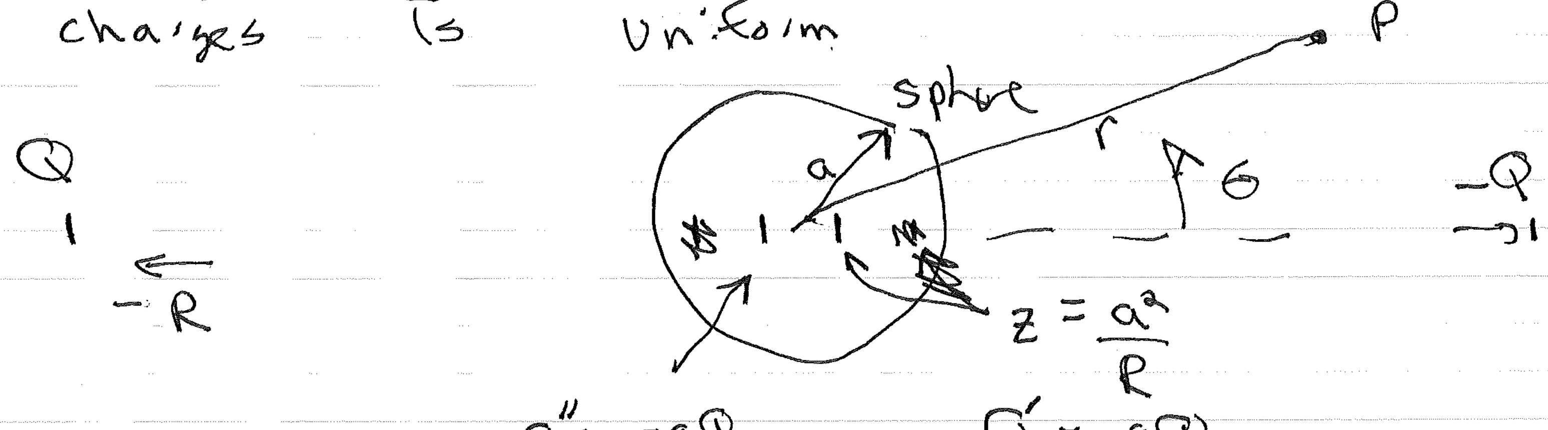
$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} - \frac{a}{y}\frac{q}{\left|\vec{x}-\frac{a^2}{y^2}\vec{y}\right|}\right] + \frac{Va}{|\vec{x}|}$$

Conducting sphere in a uniform E field



$E_0 \approx \frac{2Q}{4\pi\epsilon_0 R^2}$ parallel to z axis

as $R \to \infty$ E field from two charges is uniform



$Q'' = -\frac{aQ}{R}$ $\qquad$ $Q' = \frac{aQ}{R}$

~~Consider~~

$$4\pi\epsilon_0 \Phi = \frac{Q}{\left(r^2+R^2+2rR\cos\theta\right)^{1/2}} - \frac{Q}{\left(r^2+R^2-2rR\cos\theta\right)^{1/2}}$$

$$- \frac{aQ}{R\left(r^2+\frac{a^4}{R^2}+\frac{2a^2r}{R}\cos\theta\right)^{1/2}} + \frac{aQ}{R\left(r^2+\frac{a^4}{R^2}-\frac{2a^2r}{R}\cos\theta\right)^{1/2}}$$

expand and work to order $\frac{r^2}{R^2}$ as $R \to \infty$

$$4\pi\epsilon_0 \Phi = -\frac{2Q}{R^2} r\cos\theta + \frac{2Q}{R^2}\frac{a^3}{r^2}\cos\theta + O\left(\frac{1}{R^3}\right)$$

In limit $R \to \infty$ $\quad E_0 = \frac{2Q}{4\pi\epsilon_0 R^2}$

$$\boxed{\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta}$$

First term is pot. for uniform E field and 2nd term is pot. from induced charge on sphere.